LAST Name _______Korvo\\________ FIRST Name _______Sirkule______
Lab Time _______

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.

- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5” × 11” sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—including, for example, commencing work prematurely or continuing beyond the announced stop time—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because if we can’t read it, we can’t grade it.

- For each problem, limit your work to the space provided specifically for that problem. **No other work will be considered in grading your exam. No exceptions.**

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- We hope you do a **fantastic** job on this exam.
MT3.1 (45 Points) A continuous-time LTI filter \( H \) has input \( x \) and output \( y \).

The figure below shows the nonzero portion of the filter's impulse response \( h(t) \).

(a) Let \( H(\omega) \) denote the frequency response of the filter.
Determine a reasonably simple expression for \( H(\omega) \), and provide well-labeled plots of the filter's magnitude response \( |H(\omega)| \) and phase response \( \angle H(\omega) \).
You may find it useful to know that

\[
2\sin^2 \alpha = 1 - \cos 2\alpha.
\]

**Method 1:** We can think of \( H \) as the cascade of two systems.

\[
h(t) = (f \star g)(t) \quad \implies \quad H(\omega) = F(\omega) G(\omega)
\]

\[
\frac{1}{2} \uparrow \quad f(t) \downarrow \frac{1}{2} \quad \frac{1}{2} \quad t \quad \rightarrow \quad F(\omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i\omega t} dt = i \frac{2 \sin \left( \frac{\omega}{2} \right)}{\omega} \quad \Rightarrow \quad H(\omega) = i \frac{2 \sin \left( \frac{\omega}{2} \right)}{\omega}
\]

Using the \( \sin \) identity

\[
H(\omega) = \frac{i}{\omega} \left( 1 - \cos \omega \right)
\]

\[
|H(\omega)| = \frac{1 - \cos \omega}{\omega - 1}
\]

\[
H(\omega) = \frac{i}{\omega} \left( 1 - \cos \omega \right)
\]

\[
\angle H(\omega) = \text{purely imaginary}
\]

\[
\mathcal{F}\{h(t)\} = \begin{cases} 
\frac{\pi}{2} & \omega > 0, \omega + 2\pi k \\
\text{undefined} & \omega = 2\pi k, k \in \mathbb{Z} \\
\frac{\pi}{2} & \omega < 0, \omega + 2\pi k
\end{cases}
\]

- \(-\pi, -2\pi, \ldots, 0, \pi, 2\pi, \ldots\)
- \((-\pi, \pi), (0, \pi), (\pi, 2\pi), \ldots\)
- \((-\pi, 0, \pi, 2\pi), (0, \pi), (\pi, 2\pi), \ldots\)
(b) Suppose the input signal \( x \) satisfies the constraint \( x(t+1) = x(t) \), for all \( t \). Determine the output signal \( y \) completely.

**Method 1:** \( x(t+1) = x(t) \implies \text{\( x \) is periodic with period} \ P = 1 \implies \omega_{ox} = \frac{2\pi}{P} = 2\pi \implies \text{\( y(t) = x(t) \) for all } t \text{, so} \)

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{i\omega_k t} = \sum_{k=-\infty}^{\infty} X_k e^{i\pi \cdot 2\pi k t} = \sum_{k=-\infty}^{\infty} X_k H(\pi k) e^{i\pi \cdot 2\pi k t}, \quad \text{but } H(\pi k) = 0 \forall k \in \mathbb{Z}
\]

\[
\implies y(t) = 0 \quad \forall t.
\]

**Method 2:** Convolution of \( x \) with \( h \) involves subtracting one whole period of \( x \) from an adjacent whole period of \( x \), \( \implies \text{\( y(t) = 0 \) for all } t \).

(c) The signal \( v \) shown below is a periodic extension of \( h \).

\[
\text{In this case, we can express } v \text{ as follows:}
\]

\[
\forall t, \quad v(t) = \sum_{m=-\infty}^{\infty} h(t-2m).
\]

Determine \( V(\omega) \), the continuous-time Fourier transform of the signal \( v \).

**Method 1:** Think of \( v \) as the convolution of \( h \) with a periodic impulse train: \( \ldots \uparrow \uparrow (1) \uparrow \ldots \)

\[
r(t) = \begin{cases} 1 & 0 < t < 2 \\ 0 & \text{otherwise} \end{cases} \quad \implies \quad r(t) = \frac{1}{r} \sum_{k=-\infty}^{\infty} e^{i\omega_k t} \implies \quad R(\omega) = \frac{1}{2r} \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_k)
\]

\[
V(\omega) = H(\omega)R(\omega) \quad \implies \quad \text{By the sampling property of the Dirac delta, we have:}
\]

\[
V(\omega) = H(\omega) \sum_{k=-\infty}^{\infty} S(\omega - \pi k) \quad \implies \quad V(\omega) = 2i \sum_{k=\text{odd}}^{\infty} S(\omega - \pi k)
\]

Purely imaginary as we expect from an odd \( v(t) \).

**Method 2:** \( v(t) = \sum_{k=-\infty}^{\infty} V_k e^{i\omega_k t} \)

\[
V_k = \frac{1}{r} \int_{0}^{3} v(t) e^{i\omega_k t} dt \implies V_k = \frac{1}{r} \int_{0}^{3} \left( \frac{1}{3} \right) e^{i\omega_k t} dt = \frac{1}{r} \left( \frac{1}{3} \right) \left( \delta(t) \right) e^{i\omega_k t} \Delta t = \frac{1}{r} \left( \delta(t) \right) e^{i\omega_k t} \Delta t = \frac{1}{r} \left( \delta(t) \right) e^{i\omega_k t} \Delta t
\]

the rest is straightforward.
MT3.2 (45 Points) Consider a continuous-time periodic signal $g$ having fundamental period $T_0$. Let $g$ have the complex exponential Fourier series expansion

$$g(t) = \sum_{k=-\infty}^{+\infty} G_k e^{ik\omega_0 t},$$

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency of $g$.

Construct a discrete-time signal $h$ by sampling $g$ every $T_s$ seconds. That is,

$$h(n) = g(nT_s),$$

where $T_s$ is the sampling period (and $\omega_s = 2\pi/T_s$ is the sampling frequency).

(a) Suppose the discrete-time signal $h$ is periodic. In particular, $h(n + N) = h(n)$ for all $n$; the positive integer $N$ is the fundamental period of $h$.

Determine the relationship that must exist between $T_0$ and $T_s$ so that $h$ is periodic with fundamental period $N$.

Determine, too—in terms of $T_0$ and $N$—the smallest value that the sampling period $T_s$ can have.

$$h(n + N) = g((n + N)T_s) = g(nT_s + NT_s) \implies \text{Must have } NT_s = mT_0 \quad \exists m \in \mathbb{Z}$$

$$h(n) = g(nT_s) \implies \frac{T_s}{T_0} = \frac{m}{N} \quad \exists m \in \mathbb{Z}$$

For $h$ to be periodic, the sampling period $T_s$ must be a rational multiple of the period $T_0$ of the continuous-time periodic signal $g$.

The smallest $T_s$ is obtained by letting $m=1$ in the equation above:

$$T_s = \frac{mT_0}{N} = \frac{T_0}{N}$$
(b) If the discrete-time signal $h$ is periodic with fundamental period $N$ (and fundamental frequency $\Omega_0 = 2\pi/N$), then it must have a DSF expansion

$$h(n) = \sum_{\ell = -\infty}^{\infty} H_\ell e^{i\Omega_0 \ell n}.$$  

Express the coefficients $H_\ell$ in terms of the coefficients $G_k$.

$$h(n) = g(nT_s) = \sum_{k=-\infty}^{\infty} G_k e^{i\omega_0 \ell n} = \sum_{\ell = -\infty}^{\infty} G_k e^{i\Omega_0 \ell n}, \text{ where } \Omega_0 = \frac{2\pi T_s}{T_0} = \frac{2\pi m N}{m=1}$$

Note: $e^{i(k+N)\Omega_0 n} = e^{i\Omega_0 n} = e^{i(k-N)\Omega_0 n}$

This suggests an insightful grouping of the $G_k$'s as follows:

- Group $G_0, G_{\pm N}, G_{\pm 2N}, \ldots$ together
- Group $G_1, G_{1\pm N}, G_{1\pm 2N}, \ldots$ together
- Group $G_k, G_{k\pm N}, \ldots$ together

$$h(n) = \sum_{\ell = -\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} G_{k+N} e^{i\Omega_0 \ell n} \right) = H_\ell = \sum_{k=-\infty}^{\infty} G_{k+N}$$

(c) Suppose the signal $h$ is constant-valued; that is, $h(n) = C$ for all $n$.

(i) Determine the smallest appropriate value of the sampling period $T_s$.

$$h(n) = C \Rightarrow N = 1 \Rightarrow T_s = \frac{mT_0}{N} = \frac{m}{1} T_0 \Rightarrow \text{ Let } m = 1 \Rightarrow T_s = T_0$$

We expect this, of course; we obtain a constant-valued signal if we sample a periodic signal uniformly once every period.

(ii) Express the constant $C$ in terms of $g(t)$. Also, express $C$ in terms of the Fourier series coefficients $G_k$.

$$C = g(nT_s) = g(0) \Rightarrow C = g(0)$$

But $g(t) = \sum_{k=-\infty}^{\infty} G_k e^{i\omega_0 t} \Rightarrow g(0) = \sum_{k=-\infty}^{\infty} G_k \Rightarrow C = \sum_{k=-\infty}^{\infty} G_k$
MT3.3 (15 Points) [Modulation Property of the DTFT] Consider two signals $f$ and $g$, each of which has a well-defined discrete-time Fourier transform (DTFT):

$$f(n) \Leftrightarrow F(\omega), \quad g(n) \Leftrightarrow G(\omega).$$

We construct a third signal $h$ by multiplying $f$ and $g$ pointwise (i.e., by modulating $f$ with $g$, or vice versa). That is,

$$h(n) = f(n) g(n) \quad \text{for all } n.$$

Prove that $H(\omega)$, the DTFT of $h$, is given by

$$H(\omega) = \frac{1}{2\pi} \int_{2\pi} F(\lambda) G(\omega - \lambda) d\lambda \triangleq \frac{1}{2\pi} (F \circledast G)(\omega). \quad (1)$$

The integral on the right-hand side of Equation 1 is called the circular convolution or periodic convolution of $F$ and $G$. Essentially, you're asked to show that

$$h(n) = f(n) g(n) \Leftrightarrow \frac{1}{2\pi} (F \circledast G)(\omega).$$

Method 1:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} f(n) g(n) e^{-i\omega n}, \quad f(n) = \frac{1}{2\pi} \int_{2\pi} F(\lambda) e^{i\lambda n} d\lambda$$

$$\Rightarrow H(\omega) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{2\pi} F(\lambda) g(n) e^{-i\omega n} d\lambda = \frac{1}{2\pi} \int_{2\pi} F(\lambda) \left( \sum_{n=-\infty}^{\infty} g(n) e^{-i(\omega - \lambda)n} \right) d\lambda$$

$$\Rightarrow H(\omega) = \frac{1}{2\pi} \int_{2\pi} F(\lambda) G(\omega - \lambda) d\lambda \triangleq \frac{1}{2\pi} (F \circledast G)(\omega)$$

Method 2:

$$h(n) = \frac{1}{2\pi} \int_{2\pi} H(\omega) e^{i\omega n} d\omega = \frac{1}{2\pi} \left[ \frac{1}{2\pi} \int_{2\pi} F(\lambda) G(\omega - \lambda) e^{i\lambda n} d\lambda \right]$$

Using Eqn (1)

$$h(n) = \frac{1}{2\pi} \int_{2\pi} F(\lambda) \left[ \frac{1}{2\pi} \int_{2\pi} G(\omega - \lambda) e^{i\lambda n} d\lambda \right] = \frac{1}{2\pi} \int_{2\pi} F(\lambda) \left[ \frac{1}{2\pi} \int_{2\pi} G(\omega) e^{i\omega \cdot i\lambda n} d\omega \right]$$

$$\Rightarrow h(n) = \frac{1}{2\pi} \int_{2\pi} F(\lambda) \left[ \frac{1}{2\pi} \int_{2\pi} G(\omega - \lambda) e^{i\omega \cdot i\lambda n} d\omega \right] = \frac{1}{2\pi} \int_{2\pi} F(\lambda) \left[ \frac{1}{2\pi} \int_{2\pi} G(\omega) e^{i\omega \cdot i\lambda n} d\omega \right]$$

$$h(n) = \frac{1}{2\pi} \int_{2\pi} F(\lambda) e^{i\lambda n} \left[ \frac{1}{2\pi} \int_{2\pi} G(\omega) e^{i\omega \cdot i\lambda n} d\omega \right] \Rightarrow h(n) = f(n) g(n)$$
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