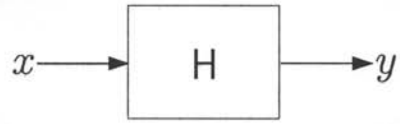
EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 3
Department of Electrical Engineering and Computer Sciences 7 May 2009
UNIVERSITY OF CALIFORNIA BERKELEY

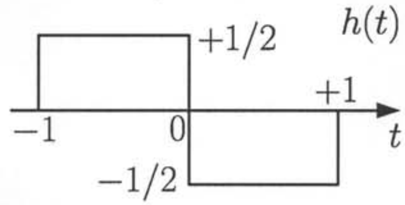
LAST Name ___Konvol___ FIRST Name ___Sirkule___
Lab Time ___?___

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT3.1 (45 Points) A continuous-time LTI filter H has input x and output y.



The figure below shows the nonzero portion of the filter's impulse response h.



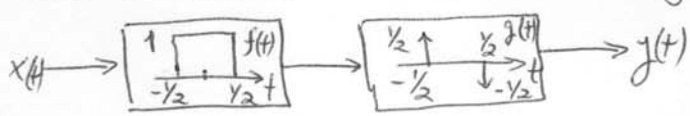
(a) Let $H(\omega)$ denote the frequency response of the filter.

Determine a reasonably simple expression for $H(\omega)$, and provide well-labeled plots of the filter's magnitude response $|H(\omega)|$ and phase response $\angle H(\omega)$.

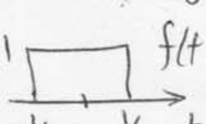
You may find it useful to know that

$$2\sin^2\alpha = 1 - \cos 2\alpha.$$

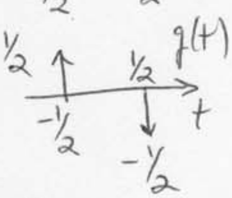
Method 1: We can think of H as the cascade of two systems



$$h(t) = (f * g)(t) \implies H(\omega) = F(\omega)\,G(\omega)$$



$$\longrightarrow F(\omega) = \int_{-1/2}^{1/2} e^{-i\omega t}\,dt = \frac{2\sin(\omega/2)}{\omega}$$



$$\longrightarrow G(\omega) = \frac{1}{2}e^{i\omega/2} - \frac{1}{2}e^{-i\omega/2} = i\sin(\omega/2)$$

$$\implies H(\omega) = i\,\frac{2\sin^2(\omega/2)}{\omega}$$

Using the trig identity $\implies$

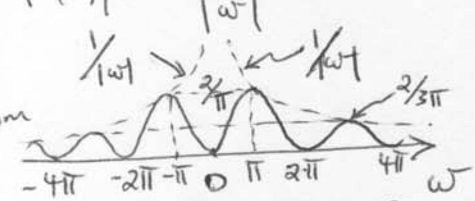
$$H(\omega) = \frac{i}{\omega}(1-\cos\omega)$$

Method 2: $$H(\omega) = \int_{-1}^{+1} h(t)e^{-i\omega t}dt = \int_{-1}^{0}\frac{e^{-i\omega t}}{2}dt - \int_{0}^{+1}\frac{e^{-i\omega t}}{2}dt = \cdots$$

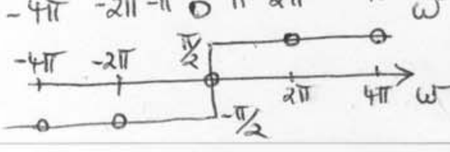
$$\implies H(\omega) = \frac{i}{\omega}(1-\cos\omega)$$

Purely imaginary as we expect from an odd $h(t)$

$$|H(\omega)| = \frac{1-\cos\omega}{|\omega|}$$



$\measuredangle H(\omega)$: $1-\cos\omega \geq 0 \implies \measuredangle H(\omega) = \begin{cases} \frac{\pi}{2} & \omega > 0,\ \omega \neq 2\pi k \\ -\frac{\pi}{2} & \omega < 0,\ \omega \neq 2\pi k \\ \text{undefined} & \omega = 2\pi k,\ k \in \mathbb{Z} \end{cases}$



(b) Suppose the input signal x satisfies the constraint $x(t+1) = x(t)$, for all t. Determine the output signal y completely.

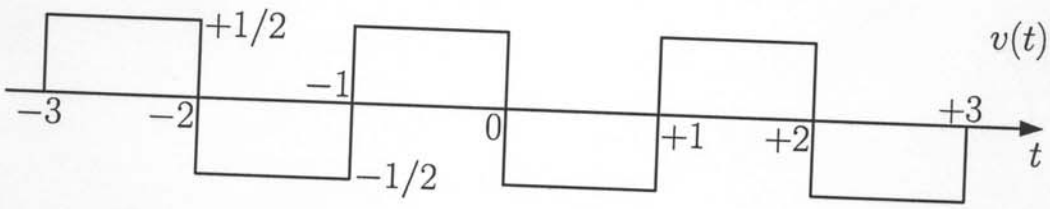
Method 1: $x(t+1)=x(t) \Rightarrow x$ is periodic w/ period $p_x=1 \Rightarrow \omega_{0x}=\frac{2\pi}{p}=2\pi \Rightarrow$

$$x(t)=\sum_{k=-\infty}^{\infty} X_k e^{ik\omega_{0x}t} = \sum_{k=-\infty}^{\infty} X_k e^{i2\pi kt} \Rightarrow y(t)=\sum_{k=-\infty}^{\infty} X_k H(2\pi k)e^{i2\pi kt}, \text{ but } H(2\pi k)=0 \ \forall k\in\mathbb{Z}$$

$\Rightarrow y(t)=0 \ \forall t.$

Method 2: Convolving x with h involves subtracting one whole period of x from an adjacent whole period of $x \Rightarrow y(t)=0 \ \forall t$

(c) The signal v shown below is a *periodic extension* of h.

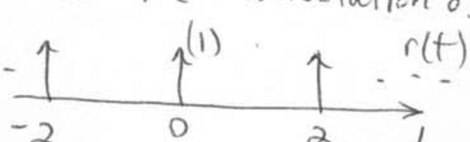


In this case, we can express v as follows:

$$\forall t, \quad v(t) = \sum_{m=-\infty}^{+\infty} h(t-2m).$$

Determine $V(\omega)$, the continuous-time Fourier transform of the signal v.

Method 1: Think of v as the convolution of h with a periodic impulse train:



$P_r=2 \rightarrow \omega_{0r}=\frac{2\pi}{P_r}=\pi$

$r(t)=\sum_{k=-\infty}^{\infty} R_k e^{ik\omega_{0r}t}$

$R_k=\frac{1}{P_r}$ ← we've done this many times $\Rightarrow r(t)=\frac{1}{2}\sum_{k=-\infty}^{\infty} e^{ik\pi t} \Rightarrow R(\omega)=\frac{\cancel{2}\pi}{\cancel{2}}\sum_{k=-\infty}^{\infty}\delta(\omega-k\pi)$

$V(\omega)=H(\omega)R(\omega) \Rightarrow$ By the sampling property of the Dirac delta, we have:

$V(\omega)=\pi\sum_{k=-\infty}^{\infty} H(k\pi)\delta(\omega-k\pi)$. But $H(k\pi)=\begin{cases}0 & k \text{ even}\\ \frac{2i}{k\pi} & k \text{ odd}\end{cases} \Rightarrow$

$V(\omega)=\cancel{\pi}\sum_{k=\text{odd}}\frac{2i}{k\cancel{\pi}}\delta(\omega-k\pi) \Rightarrow \boxed{V(\omega)=2i\sum_{k=\text{odd}}\delta(\omega-k\pi)}$ Purely imaginary as we expect from an odd $v(t)$.

Method 2: $v(t)=\sum_{k=-\infty}^{\infty} V_k e^{ik\omega_{0v}t}$ $\quad P_v=2 \Rightarrow \omega_{0v}=\pi \Rightarrow v(t)=\sum_{k=-\infty}^{\infty} V_k e^{ik\pi t}$

$V_k=\frac{1}{P_v}\int_{\langle P_v\rangle} v(t)e^{-ik\omega_{0v}t}dt \Rightarrow V_k=\frac{1}{2}\int_{-1}^{1}\underbrace{v(t)}_{h(t)=v(t) \text{ for } -1<t<1}e^{-ik\pi t}dt=\frac{1}{2}\int_{-1}^{1}h(t)e^{-ik\pi t}dt=\frac{1}{2}H(k\pi)$ The rest is straightforward

MT3.2 (45 Points) Consider a continuous-time periodic signal g having fundamental period T_0. Let g have the complex exponential Fourier series expansion

$$g(t) = \sum_{k=-\infty}^{+\infty} G_k \, e^{ik\omega_0 t},$$

where $\omega_0 = 2\pi/T_0$ is the fundamental frequency of g.

Construct a discrete-time signal h by sampling g every T_s seconds. That is,

$$h(n) = g(nT_s),$$

where T_s is the sampling period (and $\omega_s = 2\pi/T_s$ is the sampling frequency).

(a) Suppose the discrete-time signal h is periodic. In particular, $h(n+N) = h(n)$ for all n; the positive integer N is the *fundamental* period of h.

Determine the relationship that must exist between T_0 and T_s so that h is periodic with fundamental period N.

Determine, too—in terms of T_0 and N—the smallest value that the sampling period T_s can have.

$$\left.\begin{array}{l} h(n+N) = g((n+N)T_s) = g(nT_s + NT_s) \\ h(n) = g(nT_s) \end{array}\right\} \Longrightarrow \text{Must have } NT_s = mT_0 \quad \exists m \in \mathbb{Z}$$

$$\Longrightarrow \underbrace{\frac{T_s}{T_0} = \frac{m}{N}}_{\exists m \in \mathbb{Z}} \Longrightarrow$$

For h to be periodic, the sampling period T_s must be a rational multiple of the period T_0 of the continuous-time periodic signal g.

The smallest T_s is obtained by letting $m=1$ in the equation above: $\underline{T_s = \left.\frac{m}{N}T_0\right|_{m=1} = \frac{T_0}{N}}$

(b) If the discrete-time signal h is periodic with fundamental period N (and fundamental frequency $\Omega_0 = 2\pi/N$), then it must have a DSF expansion

$$h(n) = \sum_{\ell=\langle N\rangle} H_\ell \, e^{i\ell\Omega_0 n}.$$

Express the coefficients H_ℓ in terms of the coefficients G_k.

$h(n) = g(nT_s) = \sum_{k=-\infty}^{\infty} G_k e^{ik\omega_0 t}\Big|_{t=nT_s} = \sum_{k=-\infty}^{\infty} G_k e^{ik\Omega_0 n}$, where $\boxed{\Omega_0 = \omega_0 T_s = \frac{2\pi T_s}{T_0} = \frac{2\pi m}{N} \quad m=1}$

Note: $e^{i(k+N)\Omega_0 n} = e^{ik\Omega_0 n} e^{iN\Omega_0 n} = e^{ik\Omega_0 n} e^{i2\pi n} = e^{ik\Omega_0 n} \Longrightarrow$

$\underline{e^{i(k+\ell N)\Omega_0 n} = e^{ik\Omega_0 n}}$ This suggests an insightful grouping of the G_k's as follows: Group $G_0, G_{\pm N}, G_{\pm 2N}, \ldots$ together

$G_1, G_{1\pm N}, G_{1\pm 2N}, \ldots$ together

... Group $G_\ell, G_{\ell \pm 2N}, \ldots$ together

$\vdots$

$G_{N-1}, G_{N-1\pm N}, G_{N-1\pm 2N}, \ldots$ together

$h(n) = \sum_{\ell=\langle N\rangle} \left(\sum_{k=-\infty}^{\infty} G_{\ell+kN} \right) e^{i\ell\Omega_0 n} \Rightarrow \underline{H_\ell = \sum_{k=-\infty}^{\infty} G_{\ell+kN}}$

(c) Suppose the signal h is constant-valued; that is, $h(n) = C$ for all n.

(i) Determine the smallest appropriate value of the sampling period T_s.

$h(n) = C \Rightarrow N = 1 \Rightarrow T_s = \frac{m T_0}{N} = \frac{m}{1} T_0 \Rightarrow$ Let $m=1 \Rightarrow \underline{T_s = T_0}$

We expect this, of course; we obtain a constant-valued signal if we sample a periodic signal uniformly once every period.

(ii) Express the constant C in terms of $g(t)$. Also, express C in terms of the Fourier series coefficients G_k?

$C = g(nT_s) = g(0) \Longrightarrow \underline{C = g(0)}$

But $g(t) = \sum_{k=-\infty}^{\infty} G_k e^{ik\omega_0 t} \Rightarrow g(0) = \sum_{k=-\infty}^{\infty} G_k \Rightarrow \underline{C = \sum_{k=-\infty}^{\infty} G_k}$

MT3.3 (15 Points) [Modulation Property of the DTFT] Consider two signals f and g, each of which has a well-defined discrete-time Fourier transform (DTFT):

$$f(n) \overset{\mathcal{F}}{\leftrightarrow} F(\omega) \qquad , \qquad g(n) \overset{\mathcal{F}}{\leftrightarrow} G(\omega).$$

We construct a third signal h by multiplying f and g pointwise (i.e., by modulating f with g, or vice versa). That is,

$$h(n) = f(n)\, g(n) \quad \text{for all } n.$$

Prove that $H(\omega)$, the DTFT of h, is given by

$$H(\omega) = \frac{1}{2\pi}\int_{\langle 2\pi \rangle} F(\lambda)\, G(\omega - \lambda)\, d\lambda \triangleq \frac{1}{2\pi}(F \circledast G)(\omega). \qquad (1)$$

The integral on the right-hand side of Equation 1 is called the *circular convolution* or *periodic convolution* of F and G. Essentially, you're asked to show that

$$h(n) = f(n)\, g(n) \quad \overset{\mathcal{F}}{\leftrightarrow} \quad \frac{1}{2\pi}(F \circledast G)(\omega).$$

As needed, feel free to reverse the order of any summation with any integral.

Method 1:

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} f(n) g(n) e^{-i\omega n}, \qquad f(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} F(\lambda) e^{+i\lambda n}\, d\lambda$$

$$\Rightarrow H(\omega) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_{\langle 2\pi\rangle} F(\lambda) e^{i\lambda n} g(n) e^{-i\omega n}\, d\lambda = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} F(\lambda)\underbrace{\left(\sum_{n=-\infty}^{\infty} g(n) e^{-i(\omega-\lambda)n}\right)}_{G(\omega-\lambda)} d\lambda$$

$$\Rightarrow H(\omega) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} F(\lambda)\, G(\omega-\lambda)\, d\lambda \triangleq \frac{1}{2\pi}(F \circledast G)(\omega)$$

Method 2:

$$h(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} H(\omega) e^{i\omega n}\, d\omega = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\omega\, e^{i\omega n} \underbrace{\left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\lambda\, F(\lambda)\, G(\omega-\lambda)\right]}_{H(\omega)} \Rightarrow$$

Using Eqn (1)

$$h(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\lambda\, F(\lambda)\left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\omega\, G(\omega-\lambda) e^{i\omega n}\right] = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\lambda\, F(\lambda)\left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\xi\, G(\xi) e^{i\xi n} e^{i\lambda n}\right] \Rightarrow$$

$$h(n) = \left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\lambda\, F(\lambda) e^{i\lambda n}\right]\left[\frac{1}{2\pi}\int_{\langle 2\pi\rangle} d\xi\, G(\xi) e^{i\xi n}\right] \xrightarrow[\Rightarrow\, \omega = \xi + \lambda]{\text{Let } \xi = \omega - \lambda \,\Rightarrow\, d\xi = d\omega} h(n) = f(n)\, g(n)$$

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

LAST Name Konvol FIRST Name Sirkule

Lab Time ?

Problem	Points	Your Score
Name	10	10
1	45	45
2	45	45
3	15	15
Total	**115**	115