LAST Name ______________________ FIRST Name ______________________
Lab Time ______________________

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.

- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.

- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5” × 11” sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—including, for example, commencing work prematurely or continuing beyond the announced stop time—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.

- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because if we can’t read it, we can’t grade it.

- For each problem, limit your work to the space provided specifically for that problem. **No other work will be considered in grading your exam. No exceptions.**

- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.

- We hope you do a **fantastic** job on this exam.
Basic Formulas:

**Discrete Fourier Series (DFS)** Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period \( p \):

\[
x(n) = \sum_{k=\langle p \rangle} X_k e^{ik\omega_0 n} \quad \leftrightarrow \quad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) e^{-ik\omega_0 n},
\]

where \( \omega_0 = \frac{2\pi}{p} \) and \( \langle p \rangle \) denotes a suitable discrete interval of length \( p \) (i.e., an interval containing \( p \) contiguous integers). For example, \( \sum_{k=\langle p \rangle} \) may denote \( \sum_{k=0}^{p-1} \) or \( \sum_{k=1}^{p} \).

**Continuous-Time Complex-Exponential Fourier Series** Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period \( p \):

\[
x(t) = \sum_{k=-\infty}^{\infty} X_k e^{ik\omega_0 t} \quad \leftrightarrow \quad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) e^{-ik\omega_0 t} dt,
\]

where \( \omega_0 = \frac{2\pi}{p} \) and \( \langle p \rangle \) denotes a suitable continuous interval of length \( p \). For example, \( \int_{\langle p \rangle} \) can denote \( \int_{0}^{p} \) or \( \int_{-p/2}^{p/2} \).

**Continuous-Time Trigonometric Fourier Series** Trigonometric Fourier series synthesis and analysis equations for a periodic continuous-time signal having period \( p \) (and fundamental frequency \( \omega_0 = \frac{2\pi}{p} \)):

\[
x(t) = A_0 + \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + \sum_{\ell=1}^{+\infty} B_\ell \sin \ell\omega_0 t
\]

\[
A_0 = \frac{1}{p} \int_{\langle p \rangle} x(t)dt.
\]

\[
A_k = \frac{2}{p} \int_{\langle p \rangle} x(t) \cos k\omega_0 t dt, \quad 1 \leq k.
\]

\[
B_\ell = \frac{2}{p} \int_{\langle p \rangle} x(t) \sin \ell\omega_0 t dt, \quad 1 \leq \ell.
\]
MT3.1 (35 Points) Consider a continuous-time system $F : \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R} \rightarrow \mathbb{R}$ having input signal $x$ and output signal $y$:

\[
\text{The system has the input-output characteristics shown below:}
\]

\[
\begin{aligned}
y(t) &= 0, & \text{if } x(t) < 0 \\
y(t) &= x(t), & \text{if } x(t) \geq 0
\end{aligned}
\]

Suppose the input to the system is a periodic sinusoid having fundamental period $p$ and characterized by

\[
\forall t \in \mathbb{R}, \quad x(t) = \sin \omega_0 t,
\]

where $\omega_0 = 2\pi/p$ is the fundamental frequency of the input sinusoid.

(a) Provide well-labeled plots of the input signal $x$ and the output signal $y$ over at least three periods. Explain why the system $F$ is aptly called a half-wave rectifier.
(b) Suppose the input sinusoid $x$ of Part (a) (i.e., $x(t) = \sin \omega_0 t$) is applied to another system $G$

\[ x \xrightarrow{G} v \]

which has the input-output characteristics shown below:

(i) Provide a well-labeled plot of at least three periods of the output signal $v$ corresponding to $x$.

(ii) Assume the output signal $y$ of system $F$ (Part (a)) corresponding to the input sinusoid $x$ has the following trigonometric Fourier series expansion:

\[
y(t) = \frac{1}{\pi} + \frac{1}{2} \sin \omega_0 t - \frac{2}{\pi} \sum_{k=2,4,6,...}^{+\infty} \frac{\cos k\omega_0 t}{k^2 - 1}.
\]

Based on what you know about $y$, determine the trigonometric Fourier series expansion for the output signal $v$ of system $G$. Hint: Notice that $v(t) = y(t) + y(t - p/2)$, and use this to advantage.
The figure below shows a discrete-time system that is variously called a *downsampler*, *sampling rate compressor*, or *decimator*.

\[ y(n) = x(nN), \]

where \( N \in \{2, 3, 4, \ldots\} \).

Suppose the input signal \( x \) is periodic and that its *fundamental* period is \( p_x \in \mathbb{N} \).

(a) Show that the output signal \( y \) is periodic. What is the maximum value that the *fundamental* period \( p_y \) of the output signal \( y \) can have? Your answer should be in terms of \( p_x \).

(b) Suppose \( p_x \) and \( N \) are coprime, i.e., their greatest common divisor \( \gcd(p_x, N) = 1 \); an example of a pair of coprime numbers is \( (p_x = 3, N = 7) \), where \( \gcd(3, 7) = 1 \).

Determine the simplest possible expression for \( p_y \). Your answer may be in terms of \( p_x, N \), neither, or both, but it should not be particular to the numerical example \( (3, 7) \) above. Do feel free, however, to let the particular example guide you to the general form of the answer.
(c) Suppose $N$ is divisible by $p_x$, i.e., there is a number $L \in \mathbb{N}$ such that $N = Lp_x$; an example is $(p_x = 2, N = 10)$. Determine $p_y$ as specifically as possible. Your answer should not be particular to the numerical example $(2, 10)$. Do feel free, however, to let the example guide you to the general form of the answer.

(d) Suppose $p_x$ is divisible by $N$, i.e., there is a number $M \in \mathbb{N}$ such that $p_x = MN$; an example is $(p_x = 8, N = 4)$. Determine $p_y$ as specifically as possible. Your answer should not be particular to the numerical example $(8, 4)$. Do feel free, however, to let the example guide you to the general form of the answer.

(e) More generally, suppose $p_x$ and $N$ are not coprime, i.e., $\gcd(p_x, N) > 1$; an example of a pair of non-coprime numbers is $(p_x = 4, N = 6)$, where $\gcd(4, 6) = 2$.

Determine a simple expression for $p_y$. Your answer may be in terms of $p_x, N$, neither, or both, but not be particular to the numerical example $(4, 6)$. Do feel free, however, to let the particular example guide you to the general form of the answer.
MT3.3 (35 Points) The Hartley Series (HS) is an alternative orthogonal function expansion for periodic signals. If \( x : \mathbb{R} \rightarrow \mathbb{R} \) is a real-valued periodic continuous-time signal having fundamental period \( p \) (and fundamental frequency \( \omega_0 = 2\pi / p \)), then its Hartley Series representation is given by:

\[
x(t) = \sum_{-\infty}^{+\infty} C_k \text{cas} k\omega_0 t,
\]

where

\[
\text{cas} k\omega_0 t = \frac{\cos k\omega_0 t + \sin k\omega_0 t}{2}
\]

is called the \( k \)th cosine and sine function. The functions \( \chi_k \) and \( \psi_k \) are the orthogonal functions in the trigonometric Fourier series expansion of \( x \), and we know the following about them:

\[
\langle \chi_0, \chi_0 \rangle = p \\
\langle \chi_k, \chi_k \rangle = \langle \psi_k, \psi_k \rangle = \frac{p}{2} \text{ where } k \neq 0. \\
\langle \chi_k, \chi_m \rangle = \langle \psi_k, \psi_m \rangle = 0 \text{ where } k \neq m. \\
\langle \chi_k, \psi_\ell \rangle = 0 \text{ where } k, \ell \in \mathbb{Z}.
\]

(a) Using the following definition for the inner product of two continuous-time signals of periodicity \( p \),

\[
\langle f, g \rangle = \int_{(p)} f(t) g^*(t) \, dt,
\]

(i) show that \( \langle \Gamma_k, \Gamma_k \rangle = p, \forall k \in \mathbb{Z} \);

(ii) and establish that the cas functions are mutually orthogonal by showing that \( \langle \Gamma_k, \Gamma_l \rangle = 0, \forall k, l \in \mathbb{Z}, \text{ and } k \neq l. \)
(b) Determine the analysis equation in which $C_k$ is expressed in terms of the signal $x$ and the $k^{\text{th}}$ cas function $\Gamma_k$.

(c) If the complex-exponential Fourier series expansion of a real-valued continuous-time periodic signal $x$ is given by

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k e^{i\omega_0 kt},$$

express the HS coefficients $C_k$ in terms of the real and imaginary parts of the coefficients $X_k$ of the complex-exponential FS expansion of $x$. 
You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.
LAST Name ____________________  FIRST Name ____________________

Lab Time ____________________

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