EECS 20n — Final Exam Solutions

[5 pts.] Problem 1

\[ y(n) = \frac{1}{4} y(n-1) + x(n) + \frac{1}{2} x(n-1) \]

This is a linear constant coefficient system that is initially at rest
\[ \rightarrow \text{LTI} . \]

[15 pts.] Question 2

(a) (10 pts.) \( q(n) = \frac{1}{2} e^{\frac{2\pi}{3} n} + \frac{1}{2} e^{-\frac{2\pi}{3} n} + e^{i\pi n} \)

\[ \text{period 3} \quad \text{period 2} \]

Overall period = 6 \[ \omega_0 = \frac{2\pi}{6} \]

\[ X_2 = X_{-2} = \frac{1}{2} \]

\[ X_3 = 1 \]

\[ X_k = 0 \quad \text{otherwise} \]

(b) (5 pts.) \( v(t) = \cos(\pi t) + \cos(t) \)

\[ \text{period 2} \quad \text{period } 2\pi \]

Overall period \( p = 2m = 2\pi n \) where \( m, n \) are integers.

No such \( m, n \) exists because 2 is rational and \( 2\pi \) is irrational,
\[ \therefore \text{not periodic} . \]
[10 pts.] Question 3

(a) (5 pts.) Yes, the system $F$ can be linear.

(b) (5 pts.) The system $F$ cannot be time-invariant, because new frequencies have been created that did not exist in $X(\omega)$.

[15 pts.] Question 4

(a) (7 pts.) Reading the signals off the diagram, we have:

$$y(n) = -5y(n-1) - y(n-1) + x(n-1) + x(n)$$

Written alternatively, the LCCDE describing the system above is:

$$y(n) + 5y(n-1) + y(n-1) = x(n) + x(n-1)$$

(b) (8 pts.) We note that if $s_i(n)$ is the output of a delay block, then the input to the delay block must be $s_i(n+1)$, as shown below:

Accordingly, we can label the original delay-adder-gain block diagram with $s_1(n+1)$ and $s_2(n+1)$.

We can now read off the diagram the expressions for $s_1(n+1)$, $s_2(n+1)$, and $y(n)$.

$$s_1(n+1) = s_2(n) - 5y(n) + x(n) \Rightarrow s_1(n+1) = -5s_1(n) + s_2(n) - 4x(n)$$

$$y(n) = s_1(n) + x(n) \Rightarrow s_2(n+1) = -s_1(n) - x(n)$$

Hence, the state-space equations are:

$$\begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} -4 \\ -1 \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + 1 \cdot x(n)$$
[25 pts.] Question 5

(a) (7 pts.) \[ F(\omega) = \sum_n f(n)e^{-i\omega n} = \frac{1}{4}e^{i\omega} + 1 + \frac{1}{4}e^{-i\omega} = 1 + \frac{1}{2}\cos \omega \]

\[ F(\omega) \in \mathbb{R} \quad \Rightarrow \quad F(\omega) = |F(\omega)| = 1 + \frac{1}{2}\cos \omega \]

\[ \Delta F(\omega) = 0 \text{ for a positive real quantity} \]

-\( \pi \) -\( \pi/2 \) 0 \( \pi/2 \) \( \pi \)

Note: \( F(\omega) \), \( |F(\omega)| \), and \( \Delta F(\omega) \) are periodic and repeat outside the \((-\pi, \pi)\) interval.

(b) (8 pts.) \( h = f * g \) for a cascade interconnection.

\[ h(n) = \sum_m f(m)g(n - m) \]

\[ h(n) = \begin{cases} 
\frac{1}{4} & n = -1, 2 \\
\frac{5}{4} & n = 0, 1 \\
0 & \text{elsewhere} 
\end{cases} \]

Sanity Check:

\( f(n) \) has a region of support of length \( N = 3 \).

\( g(n) \) has a region of support of length \( M = 2 \).

\( h(n) \) has a region of support of length \( N + M - 1 = 4 \).
(c) (10 pts.) Recognize that \( g(n) = f(n-1) \) of Part (a). Hence,

\[
Q(\omega) = F(\omega)e^{-i\omega} = \left(1 + \frac{1}{2}\cos\omega\right)e^{-i\omega} \Rightarrow
\]

\[
|Q(\omega)| = |F(\omega)| = 1 + \frac{1}{2}\cos\omega \quad \omega Q(\omega) = -\omega \quad -\pi < \omega < \pi
\]

Periodic with period \( 2\pi \)

We plot only the phase here:

For an LTI system

\[
Q(\omega) \xrightarrow{\text{LTI}} |Q(\omega_0)|\cos(\omega_0n + \theta + \omega Q(\omega_0))
\]

This can be verified by recasting \( \cos(\omega_0n + \theta) \) in terms of complex exponentials and simplifying expressions after invoking the eigenfunction property of complex exponentials. Hence, the output \( v(n) \) is:

\[
v(n) = \frac{5}{4}\cos\left(\frac{\pi n}{3} + \frac{\pi}{3}\right)
\]

\[
v(n) = \frac{5}{4}\cos\left(\frac{\pi n}{3}\right)
\]
[15 pts.] Question 6

(a) (4 pts.) The unit of $\omega_0$ is \textbf{radians/second}.

The unit of $X_k$ is \textbf{volts}.

(b) (6 pts.) The unit of $\omega$ is \textbf{radians/second}.

The unit of $d\omega$ is \textbf{radians/second}.

The unit of $X(\omega)$ is: \(\frac{X(\omega)d\omega}{2\pi}\) has unit of volt \(\Rightarrow X(\omega)\) has unit of \(\frac{\text{volt} \cdot \text{rad}}{\text{rad/ sec}} = \text{volt} \cdot \text{sec} \).

No, $X_k$ and $X(\omega)$ do not have the same unit; $X_k$ has the same unit as \(\frac{X(\omega)d\omega}{2\pi}\).

(c) (5 pts.) The unit of $\Omega$ is \textbf{radians/sample}.

The unit of $d\Omega$ is \textbf{radians/sample}.

The unit of $Y(\Omega)$ is: \(\frac{Y(\Omega)d\Omega}{2\pi}\) has unit of volt \(\Rightarrow Y(\Omega)\) has unit of \(\frac{\text{volt} \cdot \text{radians}}{\text{radians/ sample}} = \text{volt} \cdot \text{sample} \).

No, the units of $\omega$ and $\Omega$ are not the same. $\omega$ is the frequency of a CT signal \(\Rightarrow\) has unit of radians/sec. $\Omega$ is the frequency of a DT signal \(\Rightarrow\) has unit of radians/sample. Again here, $2\pi$ has unit of radians.
[20 pts.] Question 7

(a) (5 pts.)

(b) (9 pts.) No, overall state $S = \{1, 2\} \times \text{Reals}$

\[
\text{state} = (\text{mode}(n), r(n))
\]

<table>
<thead>
<tr>
<th>mode($n$)</th>
<th>$r(n)$</th>
<th>mode($n+1$)</th>
<th>$r(n+1)$</th>
<th>output $y(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$</td>
<td>r(n)</td>
<td>\leq 10$</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$</td>
<td>r(n)</td>
<td>&gt; 10$</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>r(n)</td>
<td>&lt; 1$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>$</td>
<td>r(n)</td>
<td>\geq 1$</td>
<td>2</td>
</tr>
</tbody>
</table>

(c) (6 pts.) state($n$) = (mode($n$), time in mode 2, $r(n)$)

\[
S = \{1, 2\} \times \{0, 1, 2\} \times \text{Reals}
\]
[10 pts.] Question 8

(a) (5 pts.) We sample four times every second, at equally-spaced points in time:

\[ T = 0.25 \text{ sec} \Rightarrow f_s = 4 \text{ Hz} \quad (\omega_s = 8\pi \text{ rad/sec}) \]

where \( s \) is the sampling frequency.

(b) (5 pts.) \( f_s - f_2 \) will appear as \( f_1 \Rightarrow 4 = f_2 = f_1 = 1 \Rightarrow f_2 = 3 \text{ Hz} \).

Verify: \( y(t) = \cos(6\pi t) \) evaluated at \( t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \).

Second points:

\[ y(0) = \cos(0) = 1 = x(0); \quad y\left(\frac{1}{4}\right) = \cos\left(\frac{6\pi}{4}\right) = \cos\left(\frac{3\pi}{2}\right) = 0 = x\left(\frac{1}{4}\right) \]

\[ y\left(\frac{1}{2}\right) = \cos\left(\frac{6\pi}{2}\right) = \cos(3\pi) = -1 = x\left(\frac{1}{2}\right); \quad \text{and} \quad y\left(\frac{3}{4}\right) = \cos\left(\frac{6\pi \cdot 3}{4}\right) = \cos\left(\frac{9\pi}{2}\right) = 0 \]

\[ = x\left(\frac{3}{4}\right) \]
(a) (5 pts.) \( \omega_0 = 10\pi \text{ rad/sec} \Rightarrow x_1 = \frac{2}{3} F \leftrightarrow e^{i10\pi t}; \ x_{-1} = \frac{2}{3} F \leftrightarrow e^{-i10\pi t} \)

\[ x_2 = \frac{1}{3} F \leftrightarrow e^{i20\pi t}; \ x_{-2} = \frac{1}{3} F \leftrightarrow e^{-i20\pi t} \]

\[ \Rightarrow x(t) = \frac{2}{3}(e^{i10\pi t} + e^{-i10\pi t}) + \frac{1}{3}(e^{i20\pi t} + e^{-i20\pi t}) \] \[ x(t) = \frac{4}{3}\cos(10\pi t) + \frac{2}{3}\cos(20\pi t) \]

(b) (10 pts.) \( f_s = 15 \text{ Hz} \ (\omega_s = 30\pi \text{ rad/sec}) \Rightarrow X_p(\omega) = \frac{1}{T} \sum_{k} X(\omega - k\omega_s) = 15 \sum_{k} X(\omega - 30k\pi) \)

\[ H(\omega) \]

Because of the filter \( H(\omega) \), all frequency content of \( X_p(\omega) \) outside of the \((-15\pi, 15\pi)\) band will be suppressed. \( \Rightarrow \) We need only look at \( \frac{1}{T} X(\omega) \), \( \frac{1}{T} X(\omega - 30\pi) \), and \( \frac{1}{T} X(\omega + 30\pi) \):
The above three components [scaled spectral replica of $X(\omega)$] are added and only the frequency content in the $(-15\pi, 15\pi)$ band is retained (and scaled). Hence, the spectrum of $y(t)$ is:

$$\Rightarrow \ y(t) = 2\cos(10\pi t) \text{ (where } 20\pi \text{ frequency components are aliased down to } 10\pi \text{).}$$