1. **20 points** The block diagram of a feedback composition of a discrete-time system is given below:

The state \( s \), input signal \( x \) and output signal \( y \) are related by the update equation:

\[
\begin{align*}
    s(n+1) &= s(n) + x(n) \\
    y(n) &= s(n)
\end{align*}
\]

(a) **6 points** Find the zero-state impulse response of this system.

**Answer** The impulse response is

\[
\forall n \geq 0, h(n) = \begin{cases} 
0, & n = 0 \\
1, & n \geq 1
\end{cases}
\]

(b) **6 points** Find the update equation for the feedback system with input signal \( r \), output signal \( y \) and state \( s \).

**Answer** We have \( x(n) = r(n) + ky(n) = x(n) + ks(n) \). So the update equation is:

\[
\begin{align*}
    s(n+1) &= [1 + k]s(n) + r(n) \\
    y(n) &= s(n)
\end{align*}
\]

(c) **8 points** Find the zero-state impulse response for the feedback composition, when the ‘gain’ \( k = -0.5 \).

**Answer** The zero-state impulse response for the feedback composition is

\[
\forall n \geq 0, h(n) = \begin{cases} 
0, & n = 0 \\
(1 + k)^{n-1} = (0.5)^{n-1}, & n \geq 1
\end{cases}
\]
2. **20 points** The figure below is a partial hybrid system description of the dome light controller of an automobile.

When someone opens the door \((u(t) = \text{open})\), the light is turned on \((v(t) = \text{on})\). After the door is closed \((u(t) = \text{close})\) for 30 seconds, the light is turned off \((v(t) = \text{off})\). Note that the door must be closed for the entire 30 seconds, before the light is turned off.

(a) **10 points** Design the transitions (including guard, action, and output) so that the system meets this specification.

(b) **10 points** Plot the output signal \(v(t)\) and the trajectory of the refinement state \(s(t)\), \(0 \leq t \leq 60\), when the input signal is as shown below.
3. **15 points** The continuous-time signal $x$ is given by ($t$ is in seconds)

$$
\forall t \in \mathbb{R}, \quad x(t) = \cos(2\pi \times 60 + \pi/4) + 2 \cos(2\pi \times 120 + \pi/8) + 3 \cos(2\pi \times 180 + \pi/12).
$$

(a) **5 points** Is $x$ periodic? If it is, what is its period?

**Answer** Yes, it is periodic. The period is 1/60 sec.

(b) **10 points** The signal $x$ is input to a LTI system whose frequency response is

$$
\forall \omega \in \mathbb{R}, \quad H(\omega) = \begin{cases} 
1, & |\omega| < 2\pi \times 150, \\
0.5, & \text{otherwise}
\end{cases}
$$

What is the output signal $y$? Is $y$ periodic? If it is, what is its period?

**Answer** The output signal is

$$
\forall t, \quad y(t) = \cos(2\pi \times 60 + \pi/4) + 2 \cos(2\pi \times 120 + \pi/8) + 1.5 \cos(2\pi \times 180 + \pi/12)
$$

Yes, $y$ is periodic. The period is 1/60 sec.
4. **25 points** A LTI system with input signal $x$ and output signal $y$ is described by the differential equation

$$\frac{dy}{dt} + 0.5y(t) = x(t), \quad t \in R.$$ 

(a) **10 points** Suppose the input signal is $\forall t, x(t) = e^{i\omega t}$, where $\omega$ is fixed. What is the output signal $y$?

**Answer** The output signal is $\forall t, y(t) = H(\omega)e^{i\omega t}$. Substitution into the differential equation gives

$$i\omega H(\omega)e^{i\omega t} + 0.5H(\omega)e^{i\omega t} = e^{i\omega t},$$

so

$$H(\omega) = \frac{1}{0.5 + i\omega}.$$ 

Hence

$$\forall t, \quad y(t) = \frac{1}{0.5 + i\omega}e^{i\omega t}.$$ 

(b) **5 points** What is the frequency response,

$$\forall \omega \in R, \quad H(\omega) =$$

**Answer**

$$H(\omega) = \frac{1}{0.5 + i\omega}.$$ 

(c) **10 points** What is the magnitude and phase of the frequency response for $\omega = 0.5$ rad/sec?

$$|H(0.5)| =$$

$$\angle H(0.5) =$$

**Answer**

$$|H(0.5)| = \left| \frac{1}{0.5 + i0.5} \right| = \sqrt{2}$$

$$\angle H(0.5) = -\frac{\pi}{4}$$
5. 20 points

(a) 10 points Consider a continuous-time system \( S : [R \rightarrow R] \rightarrow [R \rightarrow R] \)

i. Suppose 
\[ \forall x, \forall t, \quad S(x)(t) = x(t - 2). \]
Is \( S \) time-invariant? Why?
Answer: Yes, because the system is \( S = D_2 \) (delay by 2), so for all \( T \), \( D_2 \circ D_T = D_{2+T} = D_T \circ D_2 \), and the system is time-invariant.

ii. Suppose 
\[ \forall x, \forall t, \quad S(x)(t) = x(2t). \]
Is \( S \) time-invariant? Why?
Answer: No, because consider the signal \( \forall t, x(t) = t \). Then \( y(t) = S(x)(t) = 2t \), so
\[ D_T \circ S(x)(t) = D_T(y)(t) = 2(t - T). \]
And \( z(t) = D_T(x)(t) = t - T \), so
\[ S \circ D_T(x)(t) = S(z)(t) = z(2t) = 2t - T. \]
So \( D_T \circ S \neq S \circ D_T \).

(b) A discrete-time linear system produces output \( v \) when the input is the step \( u \). What is the output \( h \) when the input is the impulse \( \delta \)?