EECS 20. Final Exam Solution
May 17, 2004.

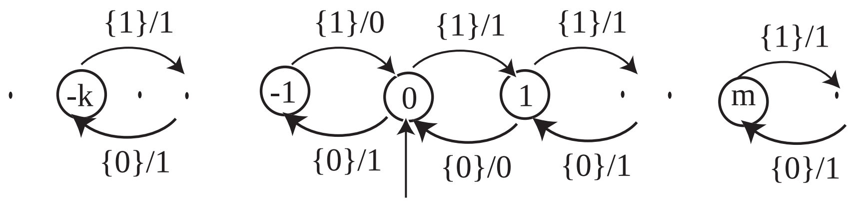
1. **10 points** A state machine has the same input and output alphabet, $\{0, 1, absent\}$.

   (a) **3 points** Its inputs signals are:

   $$InputSignals = OutputSignals = \boxed{[Nats_0 \to \{0, 1, absent\}}$$

   (b) **7 points** For any input signal x, the output signal y satisfies $y(n) = 0$ if $(x(0), ..., x(n))$ contains an *equal* number of 0's and 1's; and $y(n) = 1$, otherwise. Design a state machine (give its diagram and indicate the initial state) that has this input-ouput relationship.

   Answer This requires an infinite state machine. One machine that works is shown below. In the machine, the state $s(n)$ at any time n is the number of 1's minus the number of 0's.



2. **15 points** Consider the linear difference equation,

$$y(n) = x(n-2) + x(n-1) + x(n), \quad n \geq 0 \tag{1}$$

(a) **5 points** Give a $[A, b, c^T, d]$ representation of a state machine that satisfies this input-output relationship. What is the state $s(n)$ of your state machine in terms of x, y?

Answer Take

$$s(n) = \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix}.$$

Then

$$s(n+1) = \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(n),$$

and

$$y(n) = [1 \quad 1] \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} + [1]\, x(n)$$

from which one can read off A, b, c^T, d by matching with

$$\begin{aligned} s(n+1) &= As(n) + bx(n) \\ y(n) &= c^T s(n) + dx(n) \end{aligned}$$

(b) **3 points** What is the zero-state impulse response of this system?

Answer Taking $\forall k, x(k) = \delta(k)$, and zero initial conditions, gives the impulse response:

$$h(n) = 1, n = 0, 1, 2; \quad h(n) = 0, \text{ otherwise}.$$

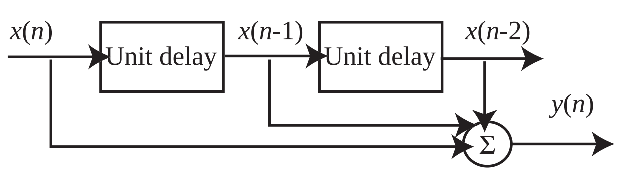
(c) **4 points** What is $y(n), n \geq 0$, if $x(-1) = x(-2) = 0$ and $x(n) = 1, n \geq 0$?

Answer Since the initial state is zero, the response is the convolution sum,

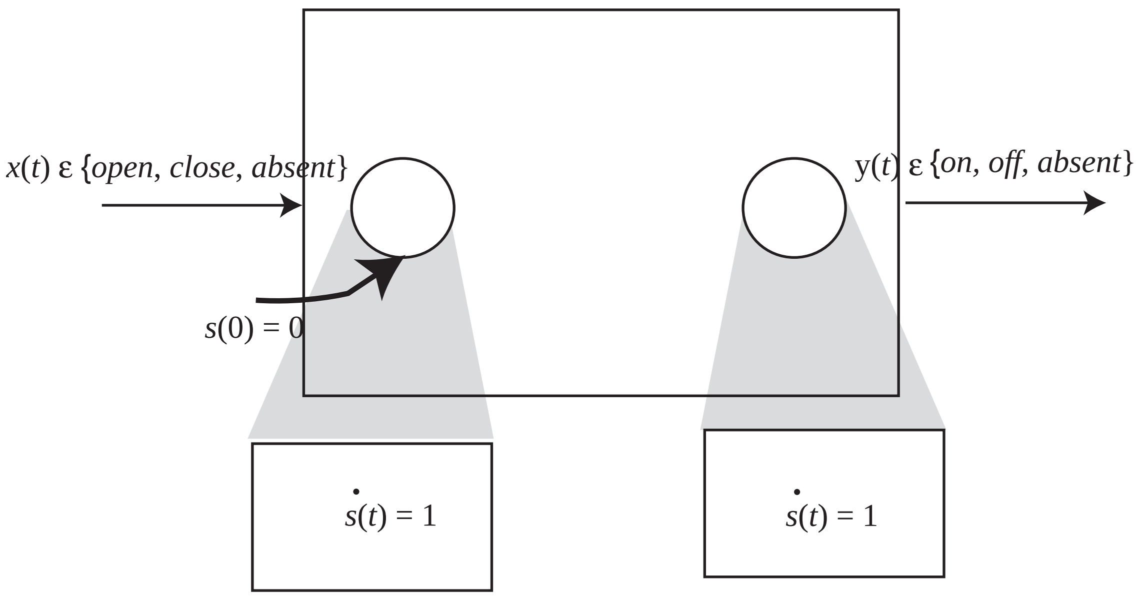
$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=0}^{2} x(n-k) = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 3, & n \geq 2 \end{cases}$$

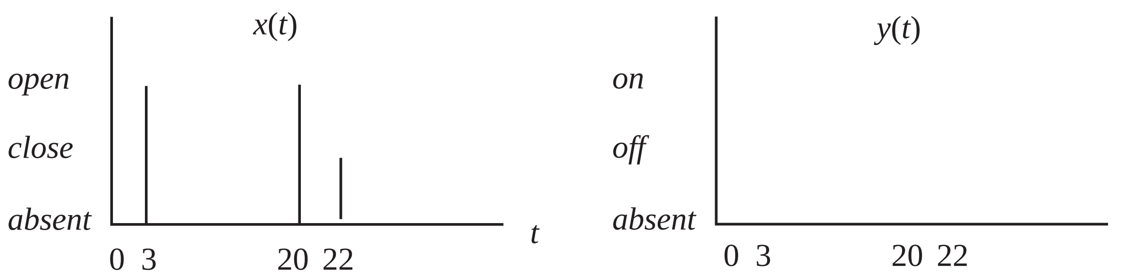
(d) **3 points** Design a tapped-delay line (give its signal flow graph) that implements (1).

Answer Two delay elements are needed and arranged as shown below.



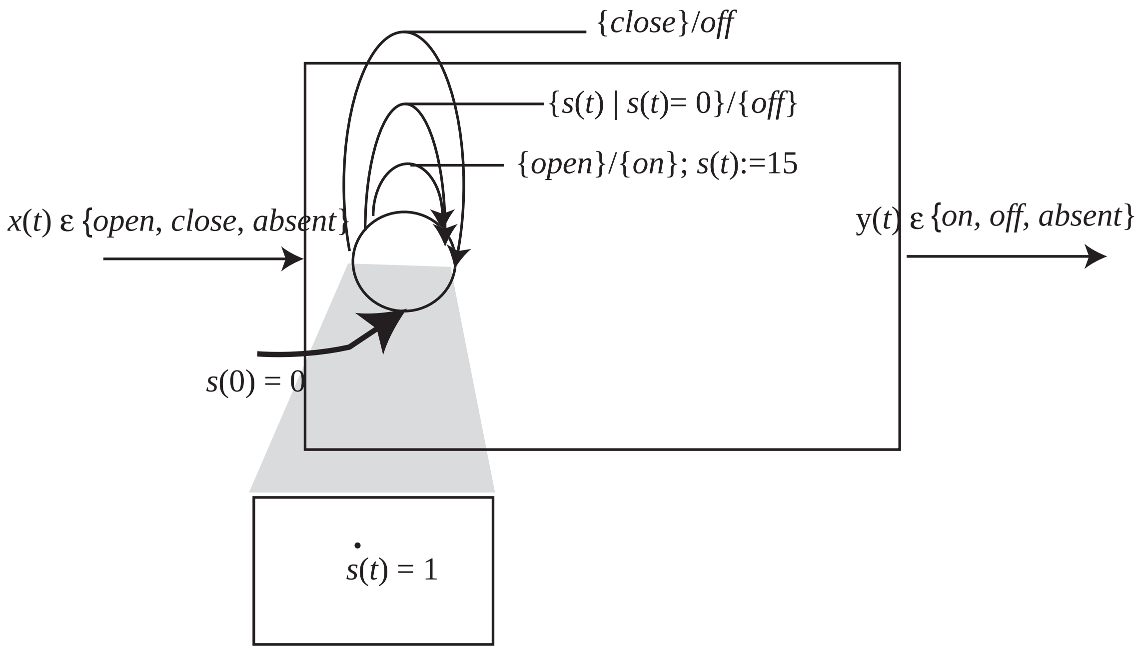
3. **10 points** The figure below is an incomplete description of a controller. When someone presses the *open* button, the output is turned *on* and 15 sec later it is turned *off*. If the *open* button is pressed before the output is *off* the output stays *on* for 15 sec beyond the last time the *open* button was pressed. If someone presses *close* while the output is *on*, it is immediately turned *off*.





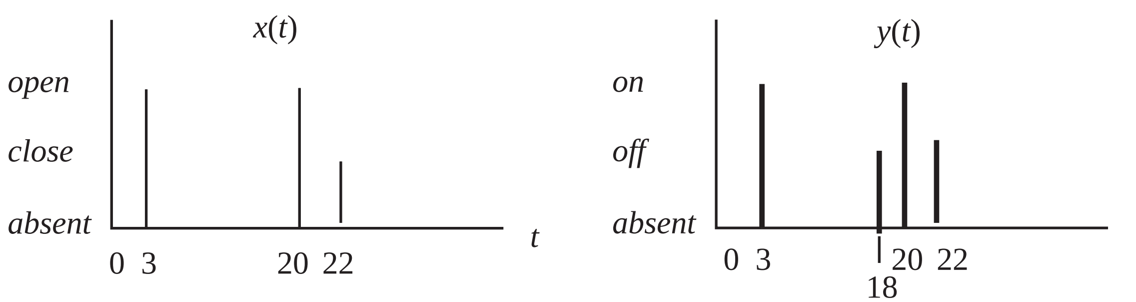
(a) **7 points** Design the guards, actions, and outputs for the transitions so as to meet this specification. Two modes are available, as shown in the figure. However, you may use only one mode.

Answer The hybrid system needs only one mode, as shown below.



(b) **3 points** Sketch the output signal y when the input signal x is as shown. Mark all time instances t when y changes value.

Answer This is shown below.



5. **15 points** Evaluate the convolution integral $y_i = h_i * x$ when x is the unit step: $x(t) = 0, t < 0; = 1, t \geq 0$, and h_i is as given below, $i = 1, 2, 3$.

(a) **5 points** $h_1(t) = 0, t < 0; = e^{-t}, t \geq 0$.

(b) **5 points** $h_2(t) = e^t, t < 0; = 0, t \geq 0$.

(c) **5 points** $h_3(t) = e^t, t < 0; = e^{-t}, t \geq 0$.

Answer In general,

$$y_i(t) = \int_{-\infty}^{\infty} h_i(s)u(t-s)ds = \int_{-\infty}^{t} h_i(s)ds.$$

Hence,

$$y_1(t) = \int_{-\infty}^{t} h_1(s)ds = \begin{cases} 0, & t \leq 0 \\ \int_0^t e^{-s}ds = 1 - e^{-t}, & t > 0 \end{cases}$$

$$y_2(t) = \int_{-\infty}^{t} h_2(s)ds = \begin{cases} \int_{-\infty}^{t} e^s ds = e^t, & t < 0 \\ \int_{-\infty}^{0} e^s ds = 1, & t \geq 0 \end{cases}$$

Since $h_3 = h_1 + h_2$, by linearity,

$$y_3(t) = y_1(t) + y_2(t) = \begin{cases} e^t, & t \leq 0 \\ 2 - e^{-t}, & t > 0 \end{cases}$$

6. **15 points** This problem concerns the various Fourier transforms.

(a) **3 points** The exponential Fourier series of the signal x,

$$\forall t \in R, \quad x(t) = \cos(2\pi t) + \sin(3\pi t),$$

is $x(t) = \sum_k X_k e^{ik\omega_0 t}$, in which $\omega_0 = \pi$, and

$$\boxed{X_{-2} = X_2 = 1/2; X_-3 = -1/(2i), X_3 = 1/(2i); X_k = 0, \text{ otherwise}}$$

(b) **5 points** The Fourier transform of the signal z,

$$\forall t \in R, \quad z(t) = e^{-t}, t \geq 0; = 0, t > 0,$$

is $\forall \omega \in R$,

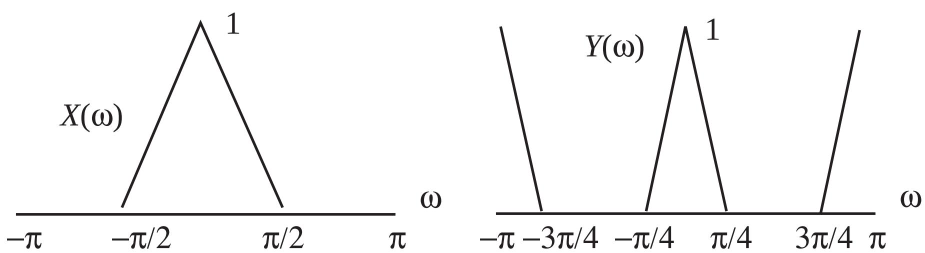
$$X(\omega) = \int_0^\infty e^{-t} e^{-i\omega t} dt = \int_0^\infty e^{-[1+i\omega]t} dt = \boxed{\frac{1}{1+i\omega}}$$

and the Fourier transform of the signal y,

$$\forall t \in R, \quad y(t) = z(t) e^{i\omega_0 t},$$

(in which z is as above) is $\boxed{\frac{1}{1+i(\omega-\omega_0)}}$

(c) **7 points** Suppose the DTFT of a signal x : *Ints* $\to$ *Complex* is as shown below.



i. Prove that the signal x is real-valued.
Proof Observe from the figure that X is real-valued and even function, so $X(\omega) = X(-\omega)^*$. Now $\forall n$,

$$\begin{aligned}[x(n)]^* &= \frac{1}{2\pi}\int_0^{2\pi} [X(\omega)e^{in\omega}]^* d\omega \\ &= \frac{1}{2\pi}\int_0^{2\pi} X(-\omega)e^{-in\omega} d\omega = \frac{1}{2\pi}\int_0^{2\pi} X(u)e^{inu} du = x(n)\end{aligned}$$

ii. Suppose the signal y is constructed by: $y(k) = x(k/2)$, if k is even; and $y(k) = 0$, if k is odd. What is the DTFT Y of y in terms of X, and sketch Y above.
Answer$Y(\omega) = \sum_k y(k)e^{ik\omega} = \sum_n x(n)e^{i2n\omega} = \boxed{X(2\omega)}$. The plot is shown above.

7. **20 points** In the figure on the next page, the left column shows three time signals, $x, p, y \in$ *ContSignals*.

(a) **5 points** Write down expressions for the corresponsing Fourier Transforms X, P, Y.

$$
\begin{aligned}
X(\omega) &= \pi[\delta(\omega - 20\pi) + \delta(\omega + 20\pi)] \\
P(\omega) &= \int_{-10}^{10} e^{-i\omega t} dt = 2\frac{\sin 10\omega}{\omega} \\
Y(\omega) &= \frac{1}{2\pi}(X * P)(\omega) = \frac{\sin 10(\omega - 20\pi)}{\omega - 20\pi} + \frac{\sin 10(\omega + 20\pi)}{\omega + 20\pi}
\end{aligned}
$$

(b) **5 points** Plot these Fourier Transforms in the column on the right. Mark the values and the frequencies at which the Fourier Transform is not zero.

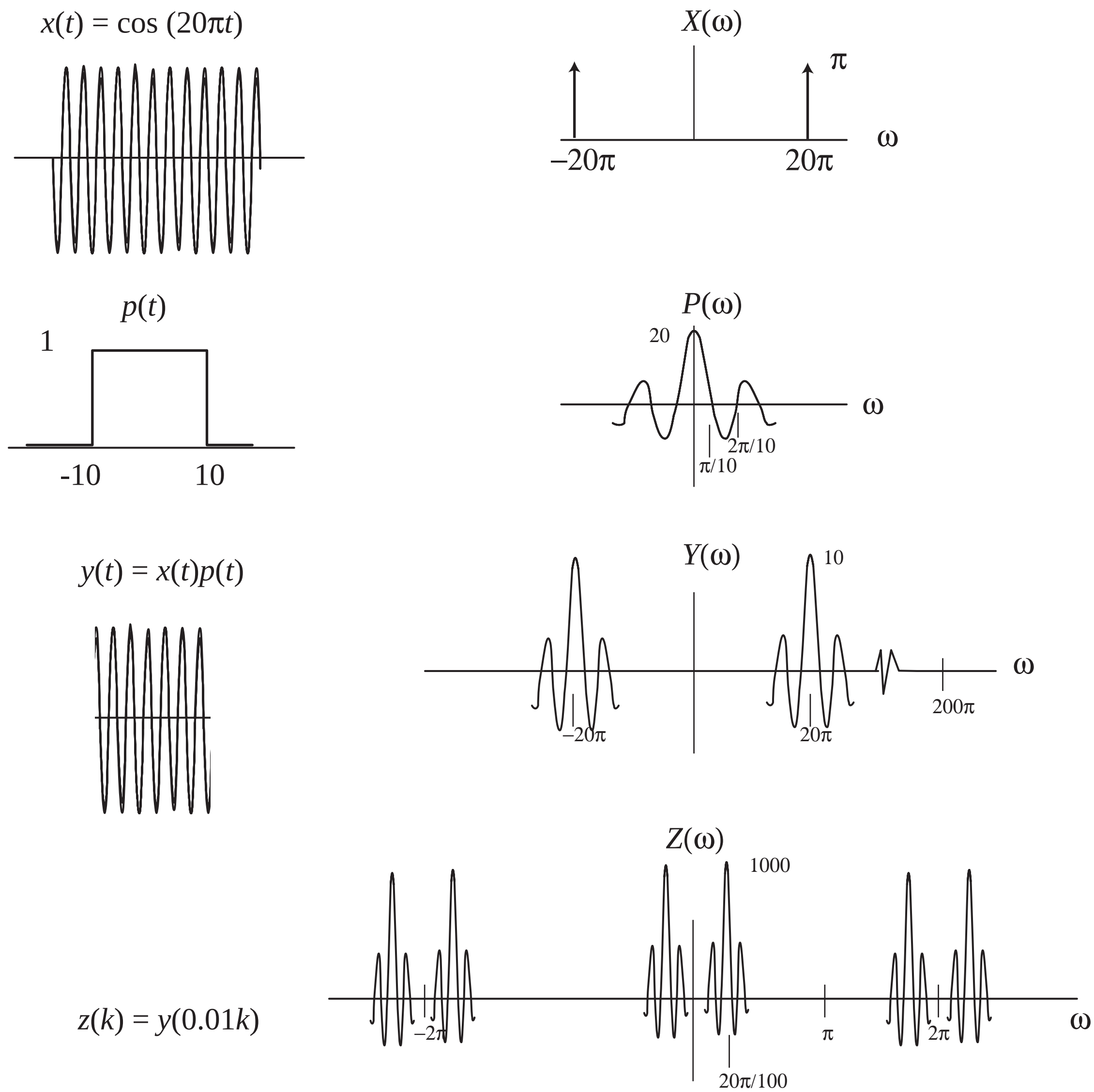
Answer See plots in figure.

(c) **5 points** Suppose the signal y is sampled every 0.01 s, i.e. the sampling frequency is 100 Hz. The sampled signal is called $z \in$ *DiscSignals*. Write down an expression for the DTFT Z of z in terms of Y.

$$
\begin{aligned}
Z(\omega) &= \frac{1}{T}\sum_k Y(\frac{\omega - 2\pi k}{T}) = 100\sum_k Y(100(\omega - 2\pi k)) \\
&= 100\sum_k \left[\frac{\sin 10(100(\omega - 2\pi k) - 20\pi)}{100(\omega - 2\pi k) - 20\pi} + \frac{\sin 10(100(\omega - 2\pi k) + 20\pi)}{100(\omega - 2\pi k) + 20\pi}\right]
\end{aligned}
$$

(d) **5 points** Sketch a plot of Z in the figure.

Answer See figure.


x(t) = cos (20πt)
X(ω)
π
ω
–20π
20π
p(t)
1
-10
10
P(ω)
20
ω
2π/10
π/10
y(t) = x(t)p(t)
Y(ω)
10
ω
–20π
20π
200π
Z(ω)
1000
z(k) = y(0.01k)
–2π
π
2π
ω
20π/100