EECS 20. Midterm No. 2 Solution
April 5, 2000.

1. (a) The fundamental frequency is $\omega_0 = 1$. The Fourier series coefficients are $A_0 = A_1 = A_2 = 1$ and $\phi_1 = -\pi/2$, with everything else having value 0.

(b) The output will have Fourier series coefficients $A_k$ scaled by the frequency response $H(\omega)$, so $A_0 = 1$ and $A_1 = 1/2$, and all others are 0. The phases of the frequency response add to those of the input, so the output will have $\phi_1 = -\pi/2 + \pi/2 = 0$. I.e., $\phi_k = 0$ for all $k$. Thus, the output is

$$y(t) = 1 + \cos(t)/2.$$  

(c) The frequency, magnitude and phase responses of the cascade composition are

$$G(\omega) = H^2(\omega),$$
$$|G(\omega)| = |H(\omega)|^2,$$
$$\angle G(\omega) = \angle H(\omega).$$

Here is a sketch:

(d) The Fourier series coefficients of the input will now be scaled by $G(\omega)$ instead of $H(\omega)$, getting $A_0 = 1$ $A_1 = 1/4$, and $\phi_1 = -\pi/2 + \pi = \pi/2$. Thus, the output is

$$y(t) = 1 + \cos(t + \pi/2)/4 = 1 - \sin(t)/4.$$  

2. (a) The sketches are shown below:
(b) Note that
\[ \delta(n) = u(n) - u(n - 1). \]
Since the system is LTI, it must therefore be true that
\[ h(n) = y(n) - y(n - 1) = \delta(n) - \delta(n - 4). \]
This is sketched below:

3. (a) Note that
\[
\cos^2\left(\frac{\pi t}{6}\right) = \frac{(e^{i\pi t/6} + e^{-i\pi t/6})^2}{4} \\
= \frac{(e^{i2\pi t/6} + 2 + e^{-i2\pi t/6})}{4} \\
= \frac{1 + \cos(2\pi t/6)}{2}.
\]
Moreover,
\[
\sin(\pi t/6) = \cos(\pi t/6 - \pi/2).
\]
Therefore
\[
x(t) = 0.5 + \cos(\pi t/6 - \pi/2) + 0.5 \cos(2\pi t/6).
\]
(b) Using the results of part (a), \(\omega_0 = \pi/6;\ A_0 = 0.5,\ A_1 = 1,\ A_2 = 0.5,\) and \(A_k = 0\) for \(k > 2;\) and \(\phi_1 = -\pi/2,\) and \(\phi_k = 0\) for \(k > 1.\)
(c) Rewriting the result from part (a),
\[
x(t) = 0.5 + \cos(\pi t/6 - \pi/2) + 0.5 \cos(2\pi t/6)
\]
\[
= 0.5 + 0.5e^{i[\pi t/6 - \pi/2]} + 0.5e^{-i[\pi t/6 - \pi/2]} + 0.25e^{i2\pi t/6} + 0.25e^{-i2\pi t/6}.
\]
From this, we can read of the Fourier series coefficients, \(X_0 = 0.5, X_1 = 0.5e^{i\pi/2} = -j/2, X_{-1} = 0.5e^{-i\pi/2} = j/2, X_2 = X_{-2} = 0.25, \) and \(X_k = 0 \) for \(k > 2 \) or \(k < -2\).

4. (a) Note from the difference equation that what we need to remember about the past is \(y(n-1)\). Thus, define the state to be
\[
s(n) = y(n-1).
\]
(You could equally well choose to define the state to be \(-0.9y(n-1), \) among other possible choices.) Thus, this is a one-dimensional SISO system. The state update equation becomes
\[
s(n + 1) = -0.9s(n) + x(n)
\]
because \(s(n + 1) = y(n)\). Thus, \(A = -0.9, \) a \(1 \times 1\) matrix, and \(b = 1\). The output equation is
\[
y(n) = -0.9s(n) + x(n)
\]
from which we recognize \(c = -0.9\) and \(d = 1\).

(b) Let the input \(x = \delta, \) the Kronecker delta function, and note that
\[
y(n) = \begin{cases} 
0, & n < 0 \\
1, & n = 0 \\
-0.9, & n = 1 \\
0.81, & n = 2 \\
(-0.9)^n, & n > 2 
\end{cases}
\]
This can be written more compactly as \(y(n) = (-0.9)^n u(n), \) where \(u(n)\) is the unit step function.