1. 40 points. Consider the state machine below

\[
\begin{align*}
\text{Inputs} &= \{1, \text{absent}\} \quad \text{and} \quad \text{Outputs} = \{0, 1, \text{absent}\} \\
(a) \text{ Is this machine deterministic or nondeterministic?} \\
\text{Answer}: \\
\text{Deterministic.} \\
(b) \text{ Give the update table.} \\
\text{Answer}: \\
\text{The update function is given by:}

<table>
<thead>
<tr>
<th>state</th>
<th>(next state, output)</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(b, 1)</td>
<td>absent</td>
</tr>
<tr>
<td>a</td>
<td>(b, 1)</td>
<td>(a, absent)</td>
</tr>
<tr>
<td>b</td>
<td>(c, 0)</td>
<td>(b, absent)</td>
</tr>
<tr>
<td>c</td>
<td>(d, 1)</td>
<td>(c, absent)</td>
</tr>
<tr>
<td>d</td>
<td>(a, 0)</td>
<td>(d, absent)</td>
</tr>
</tbody>
</table>

(c) Find a deterministic state machine that is bisimilar to this one and has only two states. Give it as a state transition diagram by completing the diagram below:

\[
\begin{align*}
\text{Answer:}
\end{align*}
\]
(d) Give the bisimulation relation.

**Answer:**

The bisimulation relation is

\[ S = \{(a, e), (b, f), (c, e), (d, f)\}, \]

or equivalently,

\[ S' = \{(e, a), (e, b), (f, c), (f, d)\}, \]

2. **30 points.** Let \( X = \{a, b, c\} \) represent a set of circles in the following picture:

Consider the following relations, all subsets of \( X \times X \):

\[
F_0 = \{(x_1, x_2) \mid \text{there is an arc going from } x_1 \text{ to } x_2 \text{ with a } 0\}
\]

\[
F_1 = \{(x_1, x_2) \mid \text{there is an arc going from } x_1 \text{ to } x_2 \text{ with a } 1\}
\]

\[
F_{\text{and}} = \{(x_1, x_2) \mid \text{there are two arcs going from } x_1 \text{ to } x_2, \text{ one with a } 0 \text{ and one with a } 1\}
\]

\[
F_{\text{or}} = \{(x_1, x_2) \mid \text{there is an arc going from } x_1 \text{ to } x_2 \text{ with a } 0 \text{ or one with a } 1\}
\]

(a) Give the elements of the four relations.

**Answer:**

\[
F_0 = \{(a, b), (b, c), (c, a)\}
\]

\[
F_1 = \{(a, a), (b, b), (c, c)\}
\]

\[
F_{\text{and}} = \emptyset
\]

\[
F_{\text{or}} = \{(a, b), (b, c), (c, a), (a, a), (b, b), (c, c)\}
\]

(b) Which of the four relations are the graph of a function of the form \( f: X \to X \)? List all that are such a graph.

**Answer:** \( F_0 \) and \( F_1 \).
(c) Are the following assertions true or false?
\[ F_{0\text{and}1} = F_0 \cap F_1 \]
\[ F_{0\text{or}1} = F_0 \cup F_1 \]

**Answer:**
Both are true.

3. **20 points** Consider all state machines with

\[ Inputs = \{1, 2, \text{absent}\} \quad \text{and} \quad Outputs = \{1, 2, \text{absent}\} \]

\[ States = \{a, b, c, d\}. \]

Assume all these state machines stutter, as usual, when presented with the stuttering input, absent.

(a) Give a state machine \( B \) that simulates all of these state machines. You will lose points if your machine is more complicated than it needs to be.

(b) Give the simulation relation.

**Answer**

\[
\begin{align*}
\{1, 2\}/1 & \\
\{1, 2, \text{absent}\}/\text{absent} & \\
\{1, 2\}/2 &
\end{align*}
\]

The simulation relation is
\[ S = \{(a, e), (b, e), (c, e), (d, e)\}. \]

4. **30 points** Consider the functions

\[ g : Y \rightarrow \text{Reals} \quad \text{and} \quad f : \text{Nats} \rightarrow Y. \]

where \( Y \) is a set.

(a) Draw a block diagram for \((g \circ f)\), with one block for each of \( g \) and \( f \), and label the inputs and output of the blocks with the domain and range of \( g \) and \( f \).
(b) Suppose $Y$ is given by

$$Y = \{[1, \cdots, 100] \rightarrow \text{Reals}\}$$

(Thus, the function $f$ takes a natural number and returns a sequence of length 100, while the function $g$ takes a sequence of length 100 and returns a real number.)

Suppose further that $g$ is given by: for all $y \in Y$,

$$g(y) = \sum_{i=1}^{100} y(i) = y(1) + y(2) + \cdots + y(100),$$

and $f$ by: for all $x \in \text{Nats}$ and $z \in \{1, \cdots, 100\}$,

$$(f(x))(z) = \cos(2\pi z/x).$$

(Thus, $x$ gives the period of a cosine waveform, and $f$ gives 100 samples of that waveform.) Give a one-line Matlab expression that evaluates $(g \circ f)(x)$ for any $x \in \text{Nats}$. Assume the value of $x$ is already in a Matlab variable called $x$.

**Answer:**

```matlab
sum(cos(2*pi*[1:100]/x))
```

(c) Find $(g \circ f)(1)$.

**Answer:** 100