
1. 20 points Fill in the blanks:

(a) If \( A = \{1, 2, 3\}, B = \{2, 3, *, #\} \), then \( A \cap B = \{2, 3\} \) and \( A \cup B = \{1, 2, 3, *, #\} \).

(b) If the predicates \( P, Q, R \) all evaluate to \( false \), then \( \neg P \land \neg Q \land \neg R \) evaluates to \( false \).

(c) If \( f : X \to Y \) and \( g : Y \to Z \), then \( g \circ f : X \to Z \).

(d) Euler’s formula is \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \).

(e) If \( A \cos(\omega t + \theta) = \cos(\omega t + \pi/4) + \cos(\omega t - \pi/4) \), then \( A = \sqrt{2}, \theta = 0 \).

2. 20 points Determine which of the following functions are periodic and what is their period in seconds or samples.

(a) \( \forall n \in \text{Ints}, \ x(n) = \cos(2\pi n/111) \). 
   Periodic, with period 111 samples.

(b) \( \forall n \in \text{Ints}, \ x(n) = \cos(2\pi \sqrt{2} n) \). 
   Not periodic.

(c) \( \forall t \in \text{Reals}, \ x(t) = \cos(2\pi \sqrt{2} t) \). 
   Periodic with period \( 1/\sqrt{2} \text{ sec} \).

(d) \( \forall t \in \text{Reals}, \ x(t) = e^{2\pi 60t + \pi/4} \). 
   Periodic with period \( 1/60 \text{ sec} \).
3. 20 points Consider a discrete-time LTI system

\[ H : [\text{Ints} \rightarrow \text{Comps}] \rightarrow [\text{Ints} \rightarrow \text{Comps}] \]

such that for input signal \( x \), the output signal \( y \) is:

\[ \forall n \in \text{Ints}, \quad y(n) = x(n) + x(n - 1). \]

(a) When the input signal \( x \) is:

\[ x(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \]

the output signal \( y \) is

\[ y(n) = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ 2, & n > 0 \end{cases} \]

(b) Obtain an expression for the the frequency response \( \hat{H}(\omega) \).

Suppose \( \forall n, x(n) = e^{i\omega n} \), then

\[ y(n) = e^{i\omega n} + e^{i\omega(n-1)} = [1 + e^{-i\omega}]e^{i\omega n} \]

so

\[ \hat{H}(\omega) = 1 + e^{-i\omega} = 1 + \cos(\omega) - i\sin(\omega) \]

(c) Expressions for the magnitude response \( |\hat{H}(\omega)| \) and the phase response \( \angle \hat{H}(\omega) \) for \( -\pi < \omega < \pi \), can be derived as follows. We have,

\[ |\hat{H}(\omega)| = \sqrt{(1 + \cos(\omega))^2 + (\sin(\omega))^2} = \sqrt{2 + 2\cos(\omega)} \]

and

\[ \angle \hat{H}(\omega) = -\tan^{-1}\left(\frac{\sin(\omega)}{1+\cos(\omega)}\right) \]

(d) Since \( \hat{H} \) is periodic with period \( 2\pi \), and since \( \hat{H}(-\omega) = (\hat{H}(\omega))^* \), we only need to plot the frequency response for \( 0 \leq \omega \leq \pi \). Here are the plots. In drawing the plots we can use the following:

\[ |\hat{H}(0)| = 2, |\hat{H}(\pi/2)| = \sqrt{2}, |\hat{H}(\pi)| = 0 \]

and

\[ \angle \hat{H}(0) = 0, \angle \hat{H}(\pi/2) = -\pi/4, \angle \hat{H}(\pi) = -\pi/2. \]
4. The exponential Fourier series of the square wave periodic function $x$ depicted in the figure is of the form:

$$
\forall t \in \text{Reals}, \quad x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(ik\omega_0 t).
$$

(a) What is $\omega_0$? $\omega_0 = 2\pi/2 = \pi$ rads/s.

(b) Calculate the coefficients $X_k$ in (1).

The general formula is

$$
X_k = \frac{1}{2} \int_{0}^{2} x(t)e^{-ik\omega_0 t} dt
= \frac{1}{2} \int_{0}^{2} x(t)e^{-ik\pi t} dt
= \frac{1}{2} \left[ \int_{0}^{1} x(t)e^{-ik\pi t} dt - \int_{1}^{2} x(t)e^{-ik\pi t} dt \right]
$$
\[ \begin{align*}
&= -\frac{1}{2\pi k} \{ e^{-ik\pi t \mid_{t=1}} - e^{-ik\pi t \mid_{t=2}} \} \\
&= -\frac{1}{2\pi k} \{ 2e^{-ik\pi} - 1 - e^{-2ik\pi} \}
\end{align*} \]

Since \( e^{-ik\pi} = 1 \) or \(-1\), according as \( k \) is even or odd, whereas \( e^{-2ik\pi} = 1 \) for all \( k \), this simplifies to:

\[ X_k = \begin{cases} 
\frac{2}{ik\pi}, & \text{if } k \text{ is odd} \\
0, & \text{if } k \text{ is even}
\end{cases} \]