EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 3
Department of Electrical Engineering and Computer Sciences 1 December 2011
University of California Berkeley

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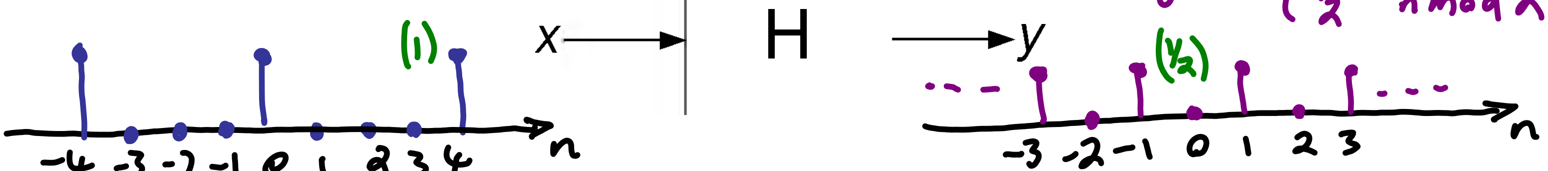
Lab Time ?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" $\times$ 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT3.1 (15 Points) Consider a discrete-time LTI system H, as shown below:

$$y(n) = \begin{cases} 0 & n \bmod 2 = 0 \\ \frac{1}{2} & n \bmod 2 \neq 0 \end{cases}$$



If the input signal is characterized by $x(n) = \sum_{\ell=-\infty}^{+\infty} \delta(n - 4\ell)$ for all n, then the corresponding output signal is

$$y(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Now, suppose we apply a different input signal to the filter—an arbitrary 2-periodic signal characterized by $x(n+2) = x(n)$ for all $n \in \mathbb{Z}$. Determine the complex-exponential discrete-time Fourier series (DTFS) expansion of the output signal y, which you must express in terms of X_k, the DTFS coefficients of the input signal x.

$$x(n) = \sum_{\ell \in \mathbb{Z}} \delta(n - 4\ell) \longrightarrow x(n) = \frac{1}{p} \sum_{k=\langle p \rangle} X_k e^{ik\omega_0 n} = \frac{1}{4} \sum_{k=0}^{3} e^{ik\frac{\pi}{2}n}$$

$$x(n) = \frac{1}{4}\left(1 + e^{i\pi n/2} + e^{i\pi n} + e^{i3\pi n/2}\right)$$

$$\boxed{\text{Freqs present: } 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}}$$

$$y(n) = \begin{cases} 0 & n \bmod 2 = 0 \\ \frac{1}{2} & n \bmod 2 \neq 0 \end{cases} \Rightarrow Y_k = \frac{1}{2} \sum_{n=0}^{1} y(n) e^{-ik\frac{2\pi}{2}n} = \frac{1}{2} y(1) e^{-ik\pi} = \frac{1}{4} e^{-ik\pi}$$

$$\Rightarrow y(n) = Y_0 + Y_1 e^{i\cdot\frac{2\pi}{2}n} = \frac{1}{4} - \frac{1}{4} e^{i\pi n}$$

Clearly $H(0) = 1$, $H(\frac{\pi}{2}) = H(\frac{3\pi}{2}) = 0$, $H(\pi) = -1$

An arbitrary 2-periodic input x has the DTFS representation $x(n) = X_0 + X_1 e^{i\pi n} \Rightarrow$ Its corresponding output is $y(n) = X_0 \underbrace{H(0)}_{=1} + X_1 \underbrace{H(\pi)}_{=-1} e^{i\pi n} = X_0 - X_1 e^{i\pi n} \Rightarrow$

$$Y_0 = X_0, \quad Y_1 = -X_1$$

MT3.2 (40 Points) Let $\mathcal{P}_2$ denote the set of all real-coefficient polynomials of degree less than, or equal to, 2, defined on the real interval $(-1, +1)$. In other words, $\mathcal{P}_2$ consists of all zeroth-order, first-order, and second-order real-coefficient polynomials defined on the interval $(-1, +1)$:

$$\mathcal{P}_2 = \left\{ at^2 + bt + c \,\middle|\, a, b, c \in \mathbb{R} \text{ and } \; t \in (-1, +1) \right\}.$$

In particular, consider the following three polynomials in $\mathcal{P}_2$:

$$\phi_0(t) = 1, \quad \phi_1(t) = \sqrt{3}\, t, \quad \text{and} \quad \phi_2(t) = \frac{\sqrt{5}}{2}\left(3t^2 - 1\right).$$

Let us define the inner product of two polynomials q and r in $\mathcal{P}_2$ as follows:

$$\langle q, r \rangle \triangleq \int_{-1}^{+1} q(t)\, r(t)\, dt.$$

(a) (18 Points) Prove the mutual orthogonality relation $\langle \phi_k, \phi_\ell \rangle = 2\,\delta(k - \ell)$, for $k, \ell = 0, 1, 2$.

$$\langle \phi_0, \phi_0 \rangle = \int_{-1}^{1} dt = 2 \qquad \langle \phi_1, \phi_1 \rangle = \int_{-1}^{1} 3t^2\, dt = 2\int_0^1 3t^2\, dt = 2\, t^3 \Big|_0^1 = 2$$

$$\langle \phi_2, \phi_2 \rangle = \frac{5}{4} \int_{-1}^{1} \left(9t^4 - 6t^2 + 1\right) dt = \frac{5}{\cancel{4}_{2}} \cdot \cancel{2} \left(\frac{9t^5}{5} - \frac{6}{3}t^3 + t\right)\Big|_0^1$$

$$\Rightarrow \langle \phi_2, \phi_2 \rangle = \frac{5}{2}\left(\frac{9}{5} - 2 + 1\right) = \frac{\cancel{5}}{\cancel{2}}\left(\frac{\cancel{4}^{\,2}}{\cancel{5}}\right) = 2$$

Each of $\langle \phi_1, \phi_0 \rangle$ and $\langle \phi_1, \phi_2 \rangle$ is zero because each of the products $\phi_1(t)\phi_0(t) = \sqrt{3}\, t$ and $\phi_1(t)\phi_2(t) = \frac{\sqrt{15}}{2}\, t(3t^2 - 1)$ is an odd function, so $\int_{-1}^{1} \phi_1(t)\phi_0(t)\, dt = \int_{-1}^{1} \phi_1(t)\phi_2(t)\, dt = 0$

As for $\langle \phi_0, \phi_2 \rangle$:

$$\langle \phi_0, \phi_2 \rangle = \frac{\sqrt{5}}{2} \int_{-1}^{1} (3t^2 - 1)\, dt = \frac{\sqrt{5}}{2}\Big[\overbrace{\int_{-1}^{1} \phi_1^2(t)\, dt}^{\langle \phi_1, \phi_1 \rangle} - \overbrace{\int_{-1}^{1} \phi_0^2(t)\, dt}^{\langle \phi_0, \phi_0 \rangle}\Big] = 0$$

(b) (17 Points) Consider an arbitrary polynomial v in $\mathcal{P}_2$ given by $v(t) = at^2+bt+c$. We can express v as a linear combination of the mutually-orthogonal polynomials ϕ_0, ϕ_1, and ϕ_2 as follows: $v(t) = \sum_{m=0}^{2} V_m\,\phi_m(t) = V_0\,\phi_0(t) + V_1\,\phi_1(t) + V_2\,\phi_2(t)$. Determine each of the expansion coefficients V_m, where $m = 0, 1, 2$, in terms the polynomial coefficients a, b, and c.

$$V_m = \frac{\langle v, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle} = \frac{1}{2}\langle v, \phi_m \rangle \qquad m = 0, 1, 2$$

odd fctn integrates to zero

$$V_0 = \frac{1}{2}\langle v, \phi_0 \rangle = \frac{1}{2}\int_{-1}^{1} (at^2 + bt + c)\,dt = \cancel{\frac{2}{2}}\int_{0}^{1} (at^2 + c)\,dt \Rightarrow V_0 = \frac{a}{3} + c$$

$$V_1 = \frac{1}{2}\langle v, \phi_1 \rangle = \frac{\sqrt{3}}{2}\int_{-1}^{1} (at^3 + bt^2 + ct)\,dt = \sqrt{3}\int_{0}^{1} bt^2\,dt \Rightarrow V_1 = \frac{b}{\sqrt{3}}$$

integrate to zero.

$$V_2 = \frac{1}{2}\langle v, \phi_2 \rangle = \frac{\sqrt{5}}{4}\int_{-1}^{1} (at^2 + bt + c)(3t^2 - 1)\,dt = \frac{\sqrt{5}}{2}\int_{0}^{1} \left[3at^4 + (3c - a)t^2 - c\right] dt$$

deleted odd powers that integrate to zero.

$$\Rightarrow V_2 = \frac{3a}{2\sqrt{5}} + \frac{\sqrt{5}}{6}(\cancel{3}c - a) - \cancel{\frac{\sqrt{5}\,c}{2}} = \frac{3\sqrt{5}}{10}a - \frac{\sqrt{5}}{6}a = \frac{2\sqrt{5}}{15}a$$

(c) (5 Points) **True or False?** If the polynomial v in part (b) is ℓ^{th}-order, where $\ell = 0, 1$, then for each $m > \ell$ the coefficient V_m must be zero. For example, if v is a first-order polynomial, then $V_2 = 0$. Similarly, if v is a zeroth-order polynomial (i.e., a constant), then $V_1 = V_2 = 0$. Provide a succinct, yet clear and convincing explanation. Depending on your approach, your answer may serve as a "sanity check" for some of the expressions you obtains in part (b).

Zeroth-Order Polynomial: $v(t) = c$ $\quad (a = b = 0)$

$V_0 = \frac{a}{3} + c = c \qquad V_1 = \frac{b}{\sqrt{3}} = 0 \qquad V_2 = \frac{2\sqrt{5}}{15}a = 0$

First-order Polynomial: $v(t) = bt + c$ $\quad (a = 0)$

$V_0 = \frac{a}{3} + c = c \qquad V_1 = \frac{b}{\sqrt{3}} \qquad V_2 = \frac{2\sqrt{5}}{15}a = 0$

Intuitive Note: ϕ_0 & ϕ_1 cannot cancel any t^2 term introduced by ϕ_2. Similarly, ϕ_0 cannot cancel any t or t^2 term introduced by ϕ_1, ϕ_2.

MT3.3 (50 Points) This problem explores aspects of the continuous-time Fourier transform (CTFT) and the LTI processing of CT periodic signals.

(a) (10 Points) Show that the Fourier transform of the continuous-time complex exponential $e^{i\omega_0 t}$ is $2\pi\,\delta(\omega-\omega_0)$. That is, establish the following CTFT pair:

$$\phi(t) = e^{i\omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \Phi(\omega) = 2\pi\,\delta(\omega-\omega_0).$$

Hint: The CTFT analysis equation will be of no help to you. Instead, think of the frequency content of the complex exponential; from that, guess the general form of its spectrum $\Phi(\omega)$; and then plug your guessed $\Phi(\omega)$ into the CTFT synthesis equation to verify its form and determine its expression precisely.

$\phi(t) = e^{i\omega_0 t}$ has only a single frequency: ω_0. So its spectrum must be of the form $\Phi(\omega) = \alpha\,\delta(\omega-\omega_0)$, where α is a scaling constant to be determined. To determine α, we use the CTFT synthesis equation:

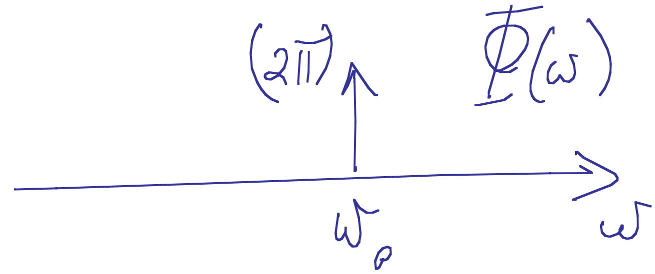
$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(\omega) e^{i\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha\,\delta(\omega-\omega_0) e^{i\omega t}\, d\omega$$

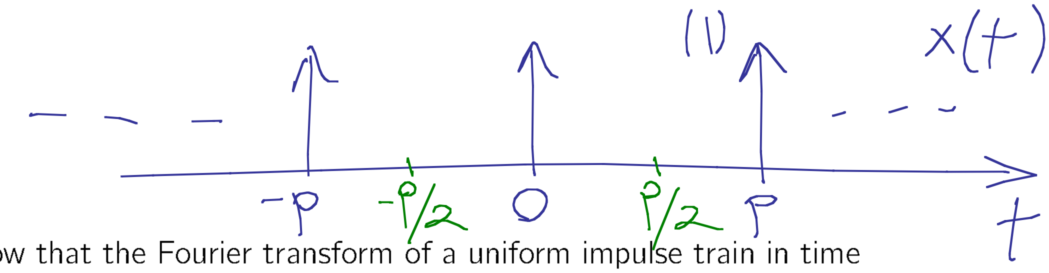
By the sifting property of the Dirac delta, we know that

$$\phi(t) = \frac{\alpha}{2\pi} e^{i\omega_0 t} \implies \alpha = 2\pi \implies$$

$$\Phi(\omega) = 2\pi\,\delta(\omega-\omega_0)$$

$$\phi(t) = e^{i\omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow}$$





(b) (15 Points) Show that the Fourier transform of a uniform impulse train in time is a uniform impulse train in frequency. In particular, show that the following constitutes a CTFT pair:

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - np) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\omega) = \omega_0 \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0),$$

where p denotes the fundamental period of the time-domain impulse train, and $\omega_0 = 2\pi/p$ represents the corresponding fundamental frequency.
Hint: Determine the continuous-time complex-exponential Fourier series representation of x, and then use the result of part (a).

$x(t) = x(t+p) \quad \forall t \Rightarrow x$ has a CTFS reperesentation:

$$x(t) = \sum_{k\in\mathbb{Z}} X_k e^{ik\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{p}$$

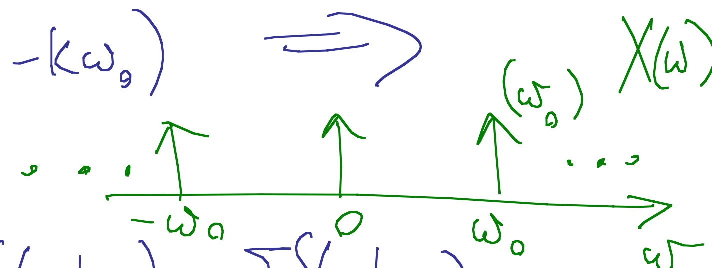
$$X_k = \frac{1}{p}\int_{\langle p\rangle} x(t) e^{-ik\omega_0 t}\,dt = \frac{1}{p}\int_{-p/2}^{p/2} \delta(t) e^{-ik\omega_0 t} = \frac{1}{p} \quad \forall k$$

$$\Rightarrow x(t) = \frac{1}{p}\sum_{k\in\mathbb{Z}} e^{ik\omega_0 t}$$

From part (a): $e^{i\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0) \Rightarrow$

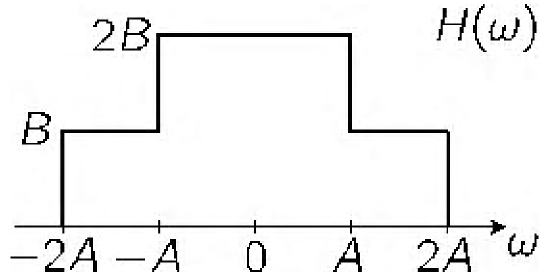
$$e^{ik\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - k\omega_0) \Rightarrow$$

$$\sum_{k\in\mathbb{Z}} e^{ik\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{k\in\mathbb{Z}} \delta(\omega - k\omega_0) \Rightarrow$$



$$x(t) = \frac{1}{p}\sum_{k\in\mathbb{Z}} e^{ik\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \frac{2\pi}{p}\sum_{k\in\mathbb{Z}} \delta(\omega - k\omega_0) = \omega_0 \sum_{k\in\mathbb{Z}} \delta(\omega - k\omega_0)$$

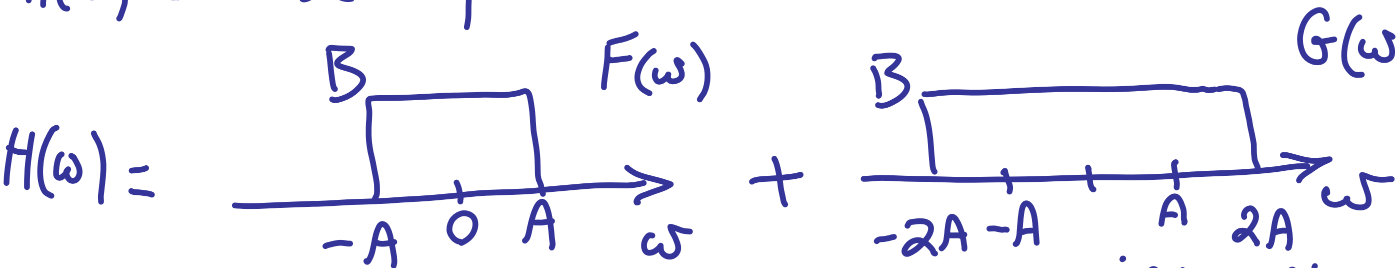
(c) (15 Points) The frequency response of a continuous-time LTI filter is shown below:



Determine a reasonably simple expression for the filter's impulse response $h(t)$.
Note: You may or may not find the trigonometric identity below useful:

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

$H(\omega)$ can be expressed as the sum of two ideal LPF spectra:



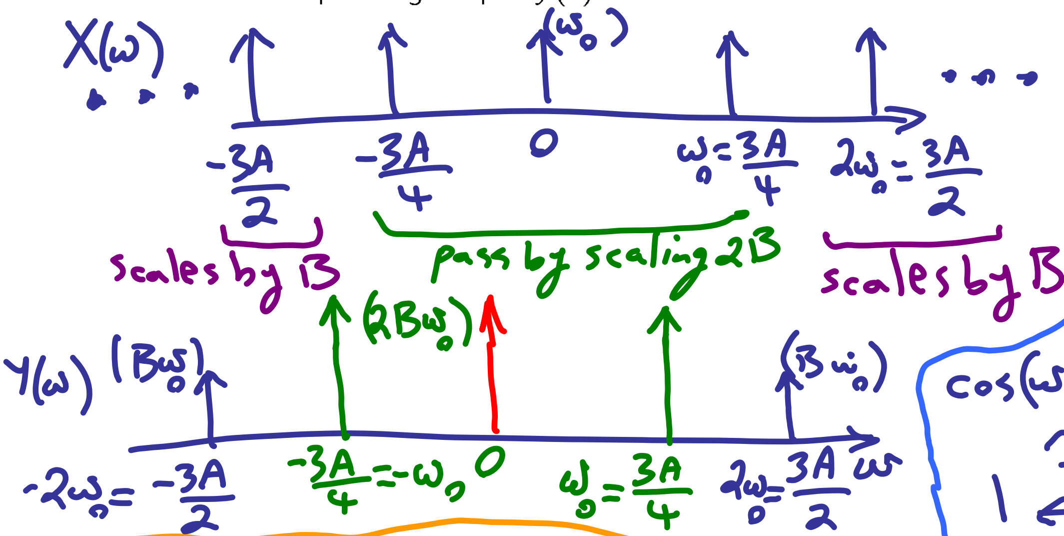
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega = \frac{B}{2\pi}\int_{-A}^{A} e^{i\omega t}d\omega = \frac{B}{2\pi}\,\frac{e^{iAt}-e^{-iAt}}{it} = \frac{B}{\pi t}\sin(At)$$

similarly, $g(t) = \frac{B}{\pi t}\sin(2At)$

$$H(\omega) = F(\omega) + G(\omega) \Rightarrow h(t) = f(t) + g(t) = \frac{B}{\pi t}\left[\sin(At) + \sin(2At)\right]$$

$$\Rightarrow h(t) = \frac{2B}{\pi t}\sin\left(\frac{3At}{2}\right)\cos\left(\frac{At}{2}\right)$$

(d) (10 Points) Apply the signal x in part (b) to the LTI system of part (c), where $p = \frac{8\pi}{3A}$ (so, $\omega_0 = \frac{3A}{4}$). Determine a reasonably simple expression for the corresponding output $y(t)$.



Only the impulses in $X(\omega)$ that survive the filter are drawn here.

Note: $e^{i\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega-\omega_0)$

$\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$

$1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$

$$y(t) = \frac{B\omega_0}{\pi} + \frac{2B\omega_0}{\pi}\cos(\omega_0 t) + \frac{B\omega_0}{\pi}\cos(2\omega_0 t)$$

$\omega_0 = \frac{3A}{4}$

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Lab Time

Problem	Points	Your Score
Name	10	10
1	15	15
2	40	40
3	50	50
Total	**115**	115

Potentially useful formulas:

DTFS:

$$x(n) = \sum_{k=\langle p\rangle} X_k\, e^{ik\omega_0 n} \quad \longleftrightarrow \quad X_k = \frac{1}{p}\sum_{n=\langle p\rangle} x(n)\, e^{-ik\omega_0 n}.$$

CTFS:

$$x(t) = \sum_{k=-\infty}^{+\infty} X_k\, e^{ik\omega_0 t} \quad \longleftrightarrow \quad X_k = \frac{1}{p}\int_{\langle p\rangle} x(t)\, e^{-ik\omega_0 t} dt.$$

DTFT:

$$x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\omega)\, e^{i\omega n} d\omega \quad \longleftrightarrow \quad X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-i\omega n}$$

CTFT:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)\, e^{i\omega t} d\omega \quad \longleftrightarrow \quad X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-i\omega t} dt$$