Solutions to MT4

EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 4
Department of Electrical Engineering and Computer Sciences 7 December 2006
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Mrofsnarf FIRST Name Reiruof

Lab Time 365/24/60/60

- **(10 Points)** Print your name and lab time in legible, block lettering above (5 points) AND on the last page (5 points) where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except four double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT4.1 (20 Points) Consider a finite-length signal f whose discrete-time Fourier transform (DTFT) F is characterized as follows:

$$\forall \omega \in \mathbb{R}, \quad F(\omega) = A(\omega)\, e^{-i2\omega},$$

where

$$A(\omega) = 3 + 4\cos(\omega) + 2\cos(2\omega).$$

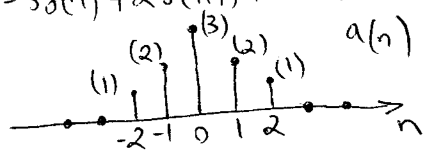
Determine the signal f completely, and provide a well-labeled stem plot of the values $f(n)$, $\forall n \in \mathbb{Z}$. Explain your work and reasoning succinctly, but clearly and convincingly.

$$f(n) \longleftrightarrow F(\omega)$$
$$a(n) \longleftrightarrow A(\omega)$$

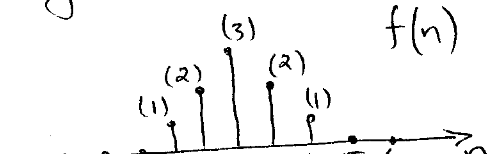
$$F(\omega) = A(\omega) e^{-i2\omega} \Longrightarrow \underline{f(n) = a(n-2)}$$

<u>Find $a(n)$:</u> $A(\omega) = 3 + 2e^{i\omega} + 2e^{-i\omega} + e^{i2\omega} + e^{-i2\omega}$

$a(n) = 3\delta(n) + 2\delta(n+1) + 2\delta(n-1) + \delta(n+2) + \delta(n-2)$



Note that $a(n) = a(-n)$ and $a(n) \in \mathbb{R}$, $\forall n$, which is consistent with the fact that $A(\omega) = A(-\omega)$, $A(\omega) \in \mathbb{R}$, $\forall \omega$.
It is now straightforward to determine $f(n)$; it is simple a two-sample delayed version of $a(n)$:



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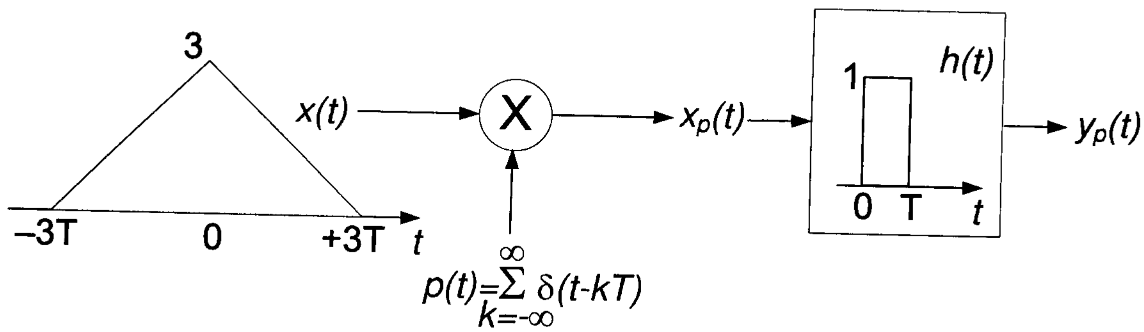
MT4.2 (~~20~~ Points) The figure below shows a signal $x : \mathbb{R} \to \mathbb{R}$ being sampled by an infinite train of ideal impulses (Dirac deltas), and then filtered by an LTI system whose rectangular impulse response $h : \mathbb{R} \to \mathbb{R}$ is characterized by

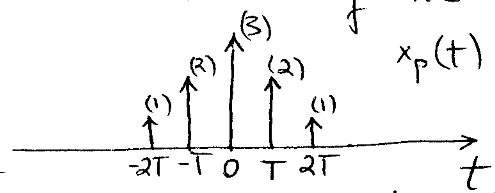
$$\forall t \in \mathbb{R}, \quad h(t) = \begin{cases} 1 & 0 \le t \le T \\ 0 & \text{elsewhere.} \end{cases}$$

The signal x and the impulse response h are zero outside the regions shown.



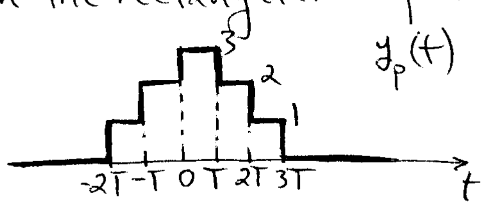
Determine, and provide a well-labeled time-domain plot for, each of the signals x_p and y_p.

x_p, which is the pointwise product of x and p, consists of a set of Dirac deltas distributed along the region of support of x. That is,

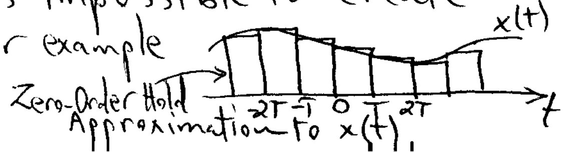


The areas (strengths) of the impulses are directly the values of x at $-2T$, $-T$, 0, T, and $2T$. This is based on the sampling Property of the Dirac delta.

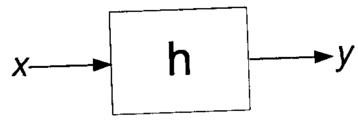
Convolving x_p with the rectangular impulse response h yields



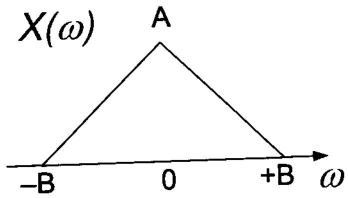
This mechanism for sampling a continuous-time signal is called "Zero-Order Hold," and it is a substitute, albeit crude, for ideal impulse train sampling, as it is impossible to create a Dirac delta in practice! Here's another example

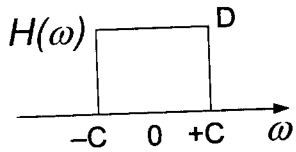


MT4.3 (45 Points) Consider the continuous-time LTI system shown below. The system has impulse response h and frequency response H.



The Fourier transform X of the input signal and the frequency response H of the LTI system are depicted below, where $A > 0$; $C = \dfrac{B}{2} > 0$; and $D > 0$.





Your answers for each part below should be in terms of the parameters A, B, C, D, or a parameter introduced in that part. Simplify all your expressions. No answer should be abandoned in the form of an integral.

(a) **(9 Points)** Determine $\displaystyle\int_{-\infty}^{+\infty} x(t)\, dt$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \Rightarrow X(0) = \int_{-\infty}^{\infty} x(t)\,dt = A \Rightarrow X(0) = A$$

$$\Rightarrow \int_{-\infty}^{\infty} x(t)\,dt = A$$

(b) **(9 Points)** Determine $x(0)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \Rightarrow x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$$

But the area under $X(\omega)$ is simply $\frac{\cancel{2}AB}{\cancel{2}} = AB \Rightarrow$

$$x(0) = \frac{AB}{2\pi}$$

For parts (c) and (d) we make use of the Parseval-Plancherel-Rayleigh Identity (PPRI) $\int_{-\infty}^{\infty} f(t) g^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) G^*(\omega)\, d\omega$, but with a caveat. We note that $X(\omega) = X(-\omega) \in \mathbb{R},\ \forall \omega \Rightarrow x(t) = x(-t) \in \mathbb{R}$.
The same goes for H and h.

(c) **(9 Points)** Determine $\int_{-\infty}^{+\infty} h^2(t)\, dt$.

$$h(t) \in \mathbb{R} \Rightarrow \int_{-\infty}^{\infty} h^2(t)\, dt = \int_{-\infty}^{\infty} |h(t)|^2\, dt$$

By the PPRI, we know $\int_{-\infty}^{\infty} |h(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) H^*(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} H^2(\omega)\, d\omega$

Hence,

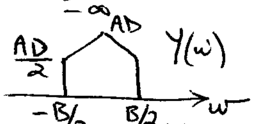
$$\int_{-\infty}^{\infty} h^2(t)\, dt = \frac{\cancel{2} C D^2}{\cancel{2}\pi} \Rightarrow \boxed{\int_{-\infty}^{\infty} h^2(t)\, dt = \frac{CD^2}{\pi}}$$

(d) **(9 Points)** Determine $\int_{-\infty}^{+\infty} x(\tau)\, h(\tau)\, d\tau$.

$$x(\tau), h(\tau) \in \mathbb{R},\ \forall \tau \in \mathbb{R} \Rightarrow \int_{-\infty}^{\infty} x(\tau) h(\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau) h^*(\tau)\, d\tau$$

By the PPRI, we have

$$\int_{-\infty}^{\infty} x(\tau) h(\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) H^*(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) H(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\, d\omega = y(0)$$



$$\Rightarrow \boxed{\int_{-\infty}^{\infty} x(\tau) h(\tau)\, d\tau = \frac{3BAD}{8\pi}}$$

Alternative: Know $h(\tau) = h(-\tau) \Rightarrow$
$\int_{-\infty}^{\infty} x(\tau) h(\tau)\, d\tau = \int_{-\infty}^{\infty} x(\tau) h(0-\tau)\, d\tau = (x * h)(0) = y(0)$

(e) **(9 Points)** For this part only, assume that the input signal is redefined to be the following:

$$\forall t \in \mathbb{R}, \quad x(t) = \cos(\omega_0 t), \quad \exists \omega_0, \text{ where } 0 < \omega_0 < C.$$

Determine a simple expression for $y(t)$, $\forall t$, values of the output signal y in the time domain.

$$y(t) = |H(\omega_0)| \cos(\omega_0 t + \measuredangle H(\omega_0))$$

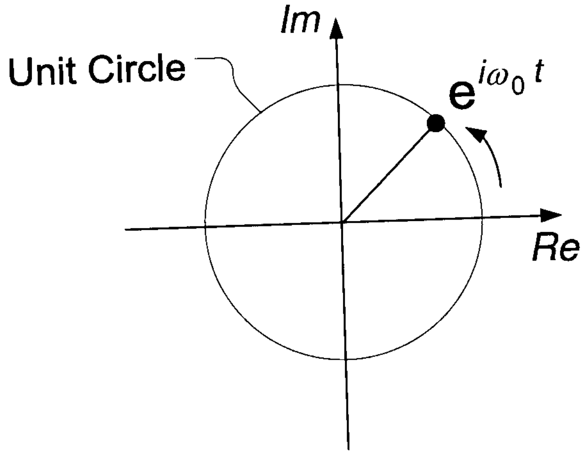
$H(\omega)$ is real and positive for $-C < \omega_0 < C \Rightarrow$

$$\boxed{y(t) = D\cos(\omega_0 t)}$$

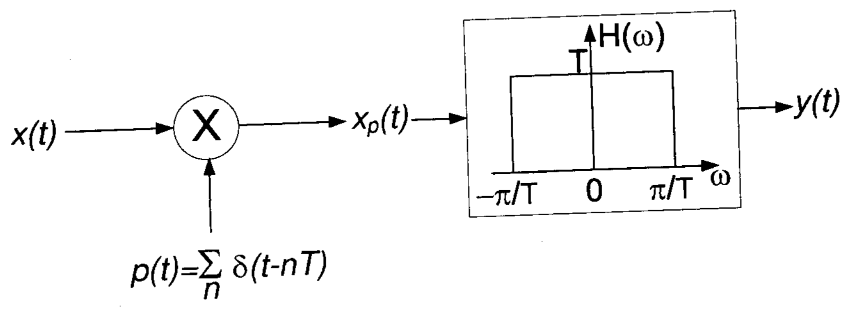
MT4.4 (25 Points) In this problem, you will discover the visual illusions created by under-sampling the rotation of a carriage wheel of unit radius.

The figure below is a model of the wheel rotating counter-clockwise at a constant rate of ω_0 radians per second. One way to model the wheel's rotation is by marking a point on the circumference of the wheel and denoting its motion using the function x, where

$$x(t) = e^{i\omega_0 t}, \forall t \in \mathbb{R}.$$



The figure below depicts a model of what happens when the wheel is filmed at a sampling rate of $\omega_s = 2\pi/T$ radians per second. The signal x is sampled by an infinite train of continuous-time impulses (each representing a frame of a video sequence) with periodicity T seconds. The resulting continuous-time signal x_p is subsequently processed by a real, continuous-time LTI filter whose frequency response H is that of the ideal low-pass filter shown in the figure (this can model our psycho-visual processing of the signal).



Suppose the sampling frequency ω_s and the rotational frequency ω_0 are related as follows: $\omega_s = \frac{5}{4}\omega_0$. Clearly, $\omega_0 < \omega_s < 2\omega_0$, so we are sampling at too low a rate.

(a) **(10 Points)** Using the modulation property of the continuous-time Fourier transform (CTFT),

$$f(t)\,g(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\,(F * G)\,(\omega),$$

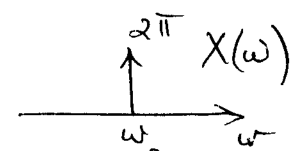
determine and provide a well-labeled plot of $X_p(\omega)$, the CTFT values of x_p. You need *not* provide a mathematical expression for X_p; you may simply draw a well-labeled plot, along with a brief explanation, to show your conceptual understanding.

The following transform pair may prove useful:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{\mathcal{F}}{\longleftrightarrow} \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s),$$

where $\omega_s = \frac{2\pi}{T}$ is the sampling frequency.

$x(t) = e^{i\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = 2\pi\,\delta(\omega - \omega_0)$



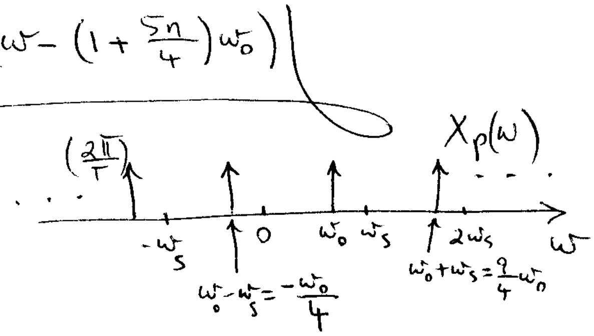
According to the modulation property, X_p is obtained essentially by convolving X and P. (modulo a scaling factor)

$$P(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

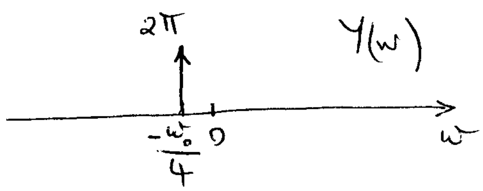
$$(X * P)(\omega) = \frac{(2\pi)^2}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 - n\omega_s) \implies X_p(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 - n\omega_s)$$

$$\omega_s = \frac{5}{4}\omega_0 \implies \underline{X_p(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \left(1 + \frac{5n}{4}\right)\omega_0\right)}$$

Plotting is the wisest way to proceed:



Low-pass filtering yields



$$\underline{Y(\omega) = 2\pi\,\delta\left(\omega + \frac{\omega_0}{4}\right)} \implies$$

$$\underline{y(t) = e^{-i\frac{\omega_0}{4}t}}$$

(b) **(10 Points)** By applying the low-pass filter to the signal x_p, determine $Y(\omega)$ and provide a well-labeled plot of the same. The signal y denotes our perception of the wheel's rotation.

Clearly, applying the low-pass filter yields $y(t) = e^{-i\left(\frac{\omega_0}{4}\right)t}$, which is a complex exponential representing a phasor that ...

1) Is moving clockwise on the unit circle, opposite the direction of the wheel's rotation; and
2) is moving at $\frac{1}{4}$ the rate of the original phasor.

(c) **(5 Points)** Determine a simple mathematical expression for $y(t)$. How does your result explain the perceived motion as being slower than, and in the opposite direction from, the actual motion of the carriage wheel?

The wheel appears to be rotating in a direction opposite that which it actually is; and it appears to be oppositely rotating at a much lower rate.

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Lab Time 365/24/60/60

Problem	Points	Your Score
Name	10	10
1	20	20
2	15	15
3	45	45
4	25	25
Total	**115**	115