EECS 20N: Structure and Interpretation of Signals and Systems Final Exam Sol.
Department of Electrical Engineering and Computer Sciences 13 December 2005
UNIVERSITY OF CALIFORNIA BERKELEY

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Lab Time 365/24/7

- **(10 Points)** Please print your name and lab time in legible, block lettering above, and on the back of the last page.
- This exam should take you about two hours to complete. However, you will be given up to about three hours to work on it. We recommend that you budget your time as a function of the point allocation and difficulty level (for you) of each problem or modular portion thereof.
- **This exam printout consists of pages numbered 1 through 14. Also included is a double-sided appendix sheet containing transform properties.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the fourteen numbered pages and the appendix. If you find a defect in your copy, notify the staff immediately.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the quiz. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, to receive full credit, you must explain your responses succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.
- It has been a pleasure having you in EECS 20N. Happy holidays!

- Complex exponential Fourier series synthesis and analysis equations for a periodic discrete-time signal having period p:

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) \, e^{-ik\omega_0 n} \ ,$$

  where $p = \frac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable contiguous discrete interval of length p (for example, $\sum_{k=\langle p \rangle}$ can denote $\sum_{k=0}^{p-1}$).

- Complex exponential Fourier series synthesis and analysis equations for a periodic continuous-time signal having period p:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \, e^{ik\omega_0 t} \qquad \longleftrightarrow \qquad X_k = \frac{1}{p} \int_{\langle p \rangle} x(t) \, e^{-ik\omega_0 t} dt \ ,$$

  where $p = \frac{2\pi}{\omega_0}$ and $\langle p \rangle$ denotes a suitable continuous interval of length p (for example, $\int_{\langle p \rangle}$ can denote $\int_0^p$).

- Discrete-time Fourier transform (DTFT) synthesis and analysis equations for a discrete-time signal:

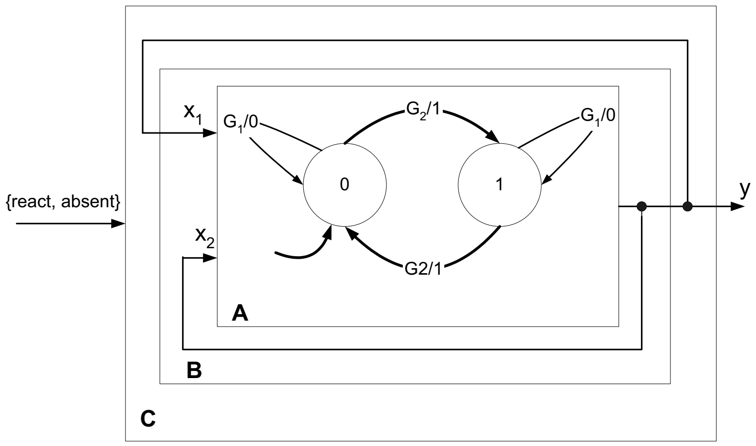
$$x(n) = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(\omega) e^{i\omega n} d\omega \qquad \longleftrightarrow \qquad X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-i\omega n} \ ,$$

  where $\langle 2\pi \rangle$ denotes a suitable continuous interval of length 2π (for example, $\int_{\langle 2\pi \rangle}$ can denote $\int_0^{2\pi}$ or $\int_{-\pi}^{\pi}$).

- Continuous-time Fourier transform (CTFT) synthesis and analysis equations for a continuous-time signal:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega \qquad \longleftrightarrow \qquad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \ .$$

F05.1 (20 Points) Consider the finite-state machine composition shown below:



Let the set $\mathsf{D} = \{0, 1, \mathsf{absent}\}$ denote an alphabet. For every pair $(x_1(n), x_2(n)) \in \mathsf{D}^2$, $x_1(n)$ and $x_2(n)$ denote the top and bottom input symbols in the figure, respectively. The n^{th} output symbol $y(n) \in \mathsf{D}$.

For *each* of the following guard sets G_1, G_2, and for each of the machines B and C, determine whether the machine is well-formed (WF) or not well-formed (NWF) by circling one choice (WF or NWF) in each entry of the table below? No explanation will be considered. No partial credit will be given.

$$(I)\begin{cases} G_1 = \{(1,0)\} \\ G_2 = \{(0,1)\} \end{cases} \qquad (II)\begin{cases} G_1 = \{(0,0),(1,0)\} \\ G_2 = \{(0,1),(1,1)\} \end{cases}$$

$$(III)\begin{cases} G_1 = \{(1,1)\} \\ G_2 = \{(0,0)\} \end{cases} \qquad (IV)\begin{cases} G_1 = \{(0,0),(1,0)\} \\ G_2 = \{\} \end{cases}$$

Guard Set	Machine B		Machine C	
(I)	[WF]	NWF	WF	[NWF]
(II)	WF	[NWF]	WF	[NWF]
(III)	WF	[NWF]	WF	[NWF]
(IV)	[WF]	NWF	[WF]	NWF

(I) C: No non-stuttering fixed point for "react." **(II)** B and C: More than one non-stuttering fixed point. **(III)** B and C: No non-stuttering fixed point.

F05.2 (40 Points) [N-Fold Upsampler] Consider a discrete-time system whose input and output signals are denoted by $x : \mathbb{Z} \to \mathbb{R}$ and $y : \mathbb{Z} \to \mathbb{R}$, respectively. The output y is obtained by upsampling the input x by a factor of N, where $N \in \{2, 3, \ldots\}$. That is,

$$\forall n \in \mathbb{Z}, \quad y(n) = \begin{cases} x\left(\dfrac{n}{N}\right) & \text{if } n \bmod N = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Select the strongest assertion from the choices below. Explain your choice.

(I) The system must be time invariant.

(II) The system could be time invariant.

[(III)] The system cannot be time invariant.

If the input signal x is defined by sample values $x(n) = \delta(n)$, the corresponding output signal y is $y(n) = \delta(n)$. Let $\widehat{x}$ denote the one-sample delay of x, i.e., $\widehat{x}(n) = x(n-1) = \delta(n-1)$. Then the corresponding output $\widehat{y}$ is characterized by $\widehat{y}(n) = \delta(n-N) \neq y(n-1) = \delta(n-1)$.

(b) Select the strongest assertion from the choices below. Explain your choice.

(I) The system must be causal.

(II) The system could be causal.

[(III)] The system cannot be causal.

The output y is such that $y(-N) = x(-1)$, which means the system peeks ahead in time.

(c) Select the strongest assertion from the choices below. Explain your choice.

(I) The system must be memoryless.

(II) The system could be memoryless.

[(III)] The system cannot be memoryless.

One approach takes advantage of the answer to part (b) by noting that a noncausal system cannot be memoryless; this is the logical contrapositive of the statement, "every memoryless system must be causal." Another approach proceeds by constructing a counterexample. Let x be defined such that $x(0) = x(1) = 1$. Then the output y is characterized by $y(0) = 1 \neq y(1) = 0$. Therefore, equal input sample values produce different output sample values, which contradicts memorylessness.

(d) Suppose the input signal x is periodic with fundamental frequency $\omega_0 = 2\pi/p$, where p denotes the period, and has the discrete Fourier series (DFS) expansion

$$x(n) = \sum_{k=\langle p \rangle} X_k \, e^{ik\omega_0 n} \, .$$

(i) Determine the period $\widehat{p}$ and the corresponding fundamental frequency $\widehat{\omega}_0$ of the periodic output signal y. Your answers must be in terms of p and ω_0.

$$\widehat{p} = pN \quad \text{and} \quad \widehat{\omega}_0 = \frac{2\pi}{\widehat{p}} = \frac{2\pi}{pN} = \frac{\omega_0}{N}.$$

(ii) Determine the DFS coefficients $Y_k, k \in \{0, 1, \ldots, \widehat{p} - 1\}$, in terms of the DFS coefficients X_k of the input signal.
Note: You can approach this problem in more than one way. Depending on which method you use, you may or may not need the following nuggets (δ denotes the Dirac delta function):

$$e^{i\omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} 2\pi \sum_{r=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi r)$$

$$\delta(\alpha(\mu - \mu_0)) = \frac{1}{|\alpha|} \delta(\mu - \mu_0) \, .$$

$$Y_k = \frac{1}{pN} \sum_{n=\langle \widehat{p} \rangle} y(n) \, e^{ik\widehat{\omega}_0 n} = \sum_{n=0}^{pN-1} y(n) \, e^{ik\widehat{\omega}_0 n}, \quad k = 0, 1, \ldots, Np - 1.$$

Method 1: At most p of the pN terms above can be nonzero: for samples $y(n)$ where n mod $N = 0$ (i.e., $n = 0, N, 2N, \ldots, (p-1)N$).

$$\begin{aligned} Y_k &= \frac{1}{pN} \left[y(0) + y(N) \, e^{ik\widehat{\omega}_0 N} + \cdots + y((p-1)N) \, e^{ik\widehat{\omega}_0 (p-1)N} \right] \\ &= \frac{1}{N} \left\{ \frac{1}{p} \left[x(0) + x(1) \, e^{ik\omega_0} + \cdots + x(p-1) \, e^{ik\omega_0 (p-1)} \right] \right\} \\ &= \frac{1}{N} X_k, \quad k = 0, 1, \ldots, p - 1. \end{aligned}$$

The complex exponentials are periodic in the index k, i.e., $e^{i(k+mp)\omega_0} = e^{ik\omega_0}, \forall m \in \mathbb{Z}$. Accordingly, the DFS coefficients X_k are periodic in k, i.e., $X_p = X_0, X_{p+1} = X_1, \ldots, X_k = X_{k \bmod p}$. Therefore,

$$\boxed{Y_k = \frac{1}{N} X_{k \bmod p}, \quad k = 0, 1, \ldots, pN - 1.}$$

Method 2: Use the CTFT impulse train expansion of periodic signals (e.g., see part (e)).

(e) Suppose $N = 2$ and $x(n) = \cos\left(\frac{\pi}{2}n\right), \forall n \in \mathbb{Z}$. Let $w : \mathbb{Z} \to \mathbb{R}$ denote the impulse response of a discrete-time low-pass LTI filter, whose frequency response is defined as follows:

$$\forall \omega \in \mathbb{R}, \quad W(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{2} \\ 0 & \text{elsewhere.} \end{cases}$$

If the upsampled signal y is processed by the LTI filter to produce the signal $v : \mathbb{Z} \to \mathbb{R}$, determine which of the following choices best characterizes v ($K \neq 0$ denotes a real constant whose value is not of concern to us here).

(I)

$$v(n) = K \cos\left(\frac{3\pi}{4}n\right), \forall n \in \mathbb{Z}.$$

(II)

$$v(n) = K \cos\left(\frac{\pi}{6}n\right), \forall n \in \mathbb{Z}.$$

(III)

$$v(n) = K \cos\left(\frac{\pi}{4}n\right), \forall n \in \mathbb{Z}.$$

(IV)

$$v(n) = K \cos\left(\frac{2\pi}{3}n\right), \forall n \in \mathbb{Z}.$$

Explain your reasoning succinctly, but clearly and convincingly.

Recall that the DTFTs of x and y are related, i.e., $Y(\omega) = X(\omega N), \ \forall \omega$. Hence,

$$\begin{aligned} X(\omega) &= \pi \sum_{r=-\infty}^{+\infty} \left[\delta\left(\omega - \frac{\pi}{2} + 2\pi r\right) + \delta\left(\omega + \frac{\pi}{2} + 2\pi r\right)\right]. \\ Y(\omega) &= X(\omega N) = \pi \sum_{r=-\infty}^{+\infty} \left[\delta\left(\omega N - \frac{\pi}{2} + 2\pi r\right) + \delta\left(\omega N + \frac{\pi}{2} + 2\pi r\right)\right] \\ &= \frac{\pi}{N} \sum_{r=-\infty}^{+\infty} \left[\delta\left(\omega - \frac{\pi}{2N} + \frac{2\pi r}{N}\right) + \delta\left(\omega + \frac{\pi}{2N} + \frac{2\pi r}{N}\right)\right]. \end{aligned}$$

With $N = 2$, only the impulses at $\pm\pi/4$ (corresponding to $r = 0$) pass through the filter. Therefore, the output is a cosine of frequency $\pi/4$. You did not have to write these expressions to get credit. You could have drawn $X(\omega)$ and $Y(\omega)$, ensuring that you would show their respective 2π- and π-periodicities, and identifying the impulses that would pass through the filter.

F05.3 **(30 Points)** Consider a finite-length continuous-time signal $x : \mathbb{R} \to \mathbb{R}$ whose region of support is confined to the interval $(-T, +T)$, where $T > 0$. That is, $x(t) = 0$, if $|t| > T$. Let $X : \mathbb{R} \to \mathbb{C}$ denote the continuous-time Fourier transform (CTFT) of x, i.e.,

$$\forall \omega \in \mathbb{R}, \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt\,.$$

Suppose the function $X(\omega)$ is modulated by the frequency-domain impulse train S, where

$$S(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\,,$$

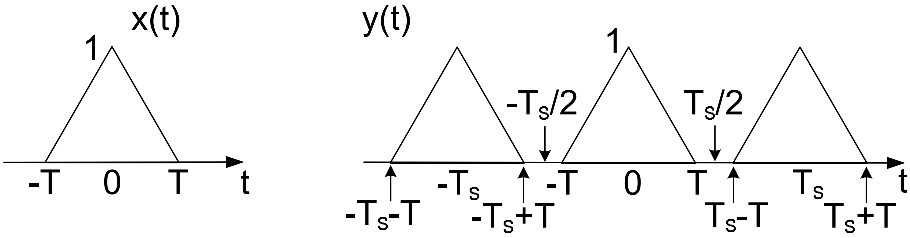
where $\omega_s = 2\pi/T_s$. Let the resulting function be denoted by Y, where $Y(\omega) = X(\omega)\, S(\omega)$. In effect, we are sampling the CTFT of x here.

(a) Determine y, the inverse Fourier transform of Y. Sketch a sample signal x and show how y is related to x.
Impulse trains in the time and frequency domains are related by:

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \overset{\mathcal{F}}{\longleftrightarrow} S(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\,.$$

Based on the convolution property, $(x * s)(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)\, S(\omega)$, we know:

$$y(t) = (x * s)(t) = \sum_{k=-\infty}^{\infty} x(t - kT_s)\,.$$



(b) What condition(s) must x satisfy so that it is recoverable from this process of frequency-domain sampling? That is, under what condition(s) (imposed on x) can we recover x from y. Explain how y should be processed to yield x.

To avoid temporal aliasing, it must be that $T_s \geq 2T$, in which case x can be recovered from y by windowing y using the window function $w : \mathbb{R} \to \mathbb{R}$, where $w(t) = 1$ for $|t| \leq T_s/2$ and $w(t) = 0$ otherwise. The figure above presumes $T_s \geq 2T$.

F05.4 (40 Points) Consider a discrete-time signal x. Each part below discloses partial information about x. Ultimately, your task is to determine x completely.

In the space provided for each part, state and explain every inference that you can draw about x, synthesizing information disclosed, or your own inferences drawn, up to, and including, that part. Justify all your work succinctly, but clearly and convincingly.

(a) The signal x coincides with, and is equal to, exactly one period of a real-valued periodic signal $\widetilde{x} : \mathbb{Z} \to \mathbb{R}$. It is known that the fundamental frequency of $\widetilde{x}$ is

$$\omega_0 = \frac{2\pi}{5} .$$

The periodic signal $\widetilde{x}$ must have period $p = 5$, because $\omega_0 \triangleq \frac{2\pi}{p}$. Therefore, x is a finite-length signal having at most five nonzero samples. We also infer that x is real-valued, i.e., $x(n) \in \mathbb{R}$, because $\widetilde{x}$—with one period of which x coincides and is equal to—is real-valued.

(b) The following is known about X, the discrete-time Fourier transform (DTFT) of x:

(i) $X(\omega) \in \mathbb{R}, \forall \omega \in \mathbb{R}$.
We infer that x must be conjugate symmetric in the time domain, i.e., $x(n) = x^*(-n), \forall n \in \mathbb{Z}$. From (a) we know that x is real-valued. Therefore, it must be that x is an even function of n, i.e., $x(n) = x(-n)$. Therefore, x is a length-5 signal centered about $n = 0$, i.e., $x(n) = 0, \forall |n| > 2$. To determine the signal x completely, we must solve for $x(-2) = x(2), x(-1) = x(1)$, and $x(0)$.

(ii) $X(\omega)\big|_{\omega=0} = \frac{3}{2}$. We infer that

$$X(0) = \sum_{n=-\infty}^{\infty} x(n) = x(0) + 2x(1) + 2x(2) = \frac{3}{2}.$$

(iii) $X(\omega)|_{\omega=\pi} = \frac{5}{2}$.

Noting that

$$X(\pi) = \sum_{n=-\infty}^{\infty} e^{i\pi n}\, x(n) = \sum_{n=-\infty}^{\infty} (-1)^n\, x(n),$$

we infer that

$$x(0) - 2x(1) + 2x(2) = \frac{5}{2}.$$

(iv) $\int_0^{2\pi} X(\omega)\, d\omega = 2\pi$.

The DTFT synthesis equation is

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)\, e^{i\omega n}\, d\omega\,.$$

We infer that

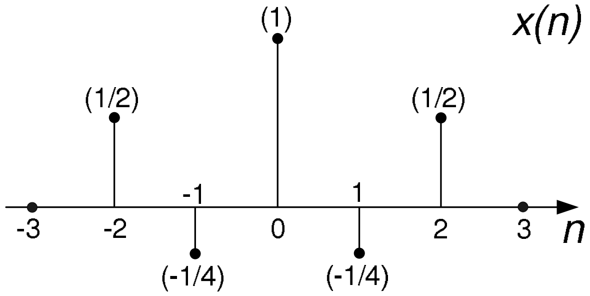
$$x(0) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)\, d\omega \; = 1.$$

- Determine, and provide a well-labeled plot of, the signal x.

We now have three equations in three unknowns:

$$\begin{cases} x(0) = 1 \\ x(0) + 2x(1) + 2x(2) = \frac{3}{2} \\ x(0) - 2x(1) + 2x(2) = \frac{5}{2} \end{cases}$$

Solving these equations for the unknown signal sample values, we find that $x(0) = 1$, $x(1) = x(-1) = -\frac{1}{4}$, $x(2) = x(-2) = \frac{1}{2}$, and $x(n) = 0$, elsewhere. The signal is plotted below:



F05.5 (40 Points) A function $f : \mathbb{R} \to \mathbb{C}$, which we call a *"mother wavelet,"* has the following properties:

1. f has zero average, i.e.,
$$\int_{-\infty}^{\infty} f(t)\, dt = 0\,.$$
2. f has finite energy, i.e.,
$$\mathcal{E}_f \triangleq ||f||^2 \triangleq \langle f, f\rangle \triangleq \int_{-\infty}^{\infty} f(t)\, f^*(t)\, dt = \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty.$$
In fact, throughout this problem, assume, without loss of generality, that f is normalized to have unit energy, i.e., $\mathcal{E}_f = 1$.

Consider a family of *"offspring wavelets"* (also called *"atoms"*) obtained by time-scaling and time-shifting f:
$$f_{\alpha,\tau}(t) = \frac{1}{\sqrt{\alpha}}\, f\left(\frac{t-\tau}{\alpha}\right)\,,$$
where $\alpha \in \mathbb{R}^+$ ($\mathbb{R}^+$ denotes the set of positive real numbers) and $\tau \in \mathbb{R}$.

(a) Suppose the mother wavelet f denotes the impulse response of a linear, time-invariant (LTI) filter. Select the strongest assertion from the choices below. Explain your reasoning succinctly, but clearly and convincingly.

(I) f could represent a low-pass filter.
(II) f must represent a low-pass filter.
(III) f could represent a band-pass filter.
[(IV)] f must represent a band-pass filter.
(V) f could represent a high-pass filter.
(VI) f must represent a high-pass filter.

The DC response of the filter is: $F(0) = \int_{-\infty}^{\infty} f(t)\, dt = 0$. Therefore, the filter cannot be low-pass. Furthermore, we can write the energy of the mother wavelet by using Parseval's relation:
$$\mathcal{E}_f = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega\,.$$
Since $\mathcal{E}_f < \infty$, it must be that $|F(\omega)| \to 0$ as $|\omega| \to \infty$, or else the integral will not be finite. A frequency response F that vanishes to zero as the frequency increases cannot be a high-pass filter. The inescapable conclusion is that the filter must be band-pass.

(b) Determine $F_{\alpha,\tau} : \mathbb{R} \to \mathbb{C}$, the continuous-time Fourier transform (CTFT) of $f_{\alpha,\tau}$. That is, determine an expression for

$$F_{\alpha,\tau}(\omega) \triangleq \int_{-\infty}^{\infty} f_{\alpha,\tau}(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} f\left(\frac{t-\tau}{\alpha}\right) e^{-i\omega t}\, dt\,.$$

Solving this problem involves a joint application of the time-shifting and time-scaling properties of the CTFT. Let $u = (t-\tau)/\alpha$, so $du = dt/\alpha$ (i.e., $dt = \alpha\, du$) and $t = \alpha u + \tau$. Noting that the limits of the integral do not change as we substitute variables (because $\alpha > 0$), we can rewrite the CTFT as follows:

$$F_{\alpha,\tau}(\omega) = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} f(u)\, e^{-i\omega(\alpha u+\tau)}\, \alpha\, du = \sqrt{\alpha}\, \underbrace{\left[\int_{-\infty}^{\infty} f(u)\, e^{-i\alpha\omega u}\, du\right]}_{F(\alpha\omega)} e^{-i\omega\tau}\,,$$

which leads to:

$$F_{\alpha,\tau}(\omega) = \sqrt{\alpha}\, F(\alpha\omega)\, e^{-i\omega\tau}.$$

(c) Consider the *Haar Family* of wavelet functions, defined by:

$$f_{2^m,n}(t) = \frac{1}{\sqrt{2^m}} f\left(\frac{t-2^m n}{2^m}\right), \quad m, n \in \mathbb{Z}\,.$$

The time-scale factor is denoted by m and the time-shift factor by n. *In what follows, assume $n = 0$.*

The mother wavelet f corresponds to $m = n = 0$, i.e.,

$$f(t) \triangleq f_{2^0,0}(t) = \begin{cases} +1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

(i) Without complicated mathematical manipulation, determine the energy and the average of each Haar atom $f_{2^m,n}$. Apply the variable substitution of part (b) to any general wavelet atom:

$$\int_{-\infty}^{\infty} f_{\alpha,\tau}(t)\, dt = \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} f\left(\frac{t-\tau}{\alpha}\right) dt = \sqrt{\alpha} \int_{-\infty}^{\infty} f(u)\, du = 0\,.$$

Similarly, the energy of every wavelet atom is:

$$\mathcal{E}_{f_{\alpha,\tau}} = \frac{1}{\alpha} \int_{-\infty}^{\infty} \left| f\left(\frac{t-\tau}{\alpha}\right)\right|^2 dt = \frac{\alpha}{\alpha} \int_{-\infty}^{\infty} |f(u)|^2\, du = 1\,.$$

The same holds for the Haar atoms.

(ii) Plotted on the next page is a sketch of the Haar mother wavelet f. In the other spaces, provide well-labeled plots of the unshifted Haar wavelet atoms characterized by $m = -1, +1, 2$, respectively.

(iii) Explain why the Haar wavelets $\{f_{2^m,0}\}_{m\in\mathbb{Z}}$ are mutually orthogonal, i.e.,

$$\langle f_{2^m,0}, f_{2^k,0}\rangle \triangleq \int_{-\infty}^{\infty} f_{2^m,0}(t)\, f^*_{2^k,0}(t)\, dt = \delta(k-m).$$

We are not looking for a rigorous mathematical proof here. You should be able to infer mutual orthogonality by observing features of the plots that you drew above and exploiting one of the salient properties of f given in the problem statement.

Consider the Haar wavelets corresponding to $m = -1$ and $m = 0$ (recall that, according to the problem statement, $n = 0$ for our purposes). Then the inner-product of $f_{2^{-1},0}$ and $f_{2^0,0}$ is:

$$\langle f_{2^{-1},0}, f_{2^0,0}\rangle = \int_0^{1/2} f_{2^{-1},0}(t)\, \underbrace{f^*_{2^0,0}(t)}_{=1}\, dt = \int_0^{1/2} f_{2^{-1},0}(t)\, dt.$$

From the corresponding plots of part (c)(ii), we note that $f_{2^0,0}$ is constant over the region of support of $f_{2^{-1},0}$. Hence, $\langle f_{2^{-1},0}, f_{2^0,0}\rangle$ is proportional to the average of $f_{2^{-1},0}$; from part (c)(i), we know that the average of each Haar wavelet is zero. The same argument holds for the inner-product of any other pair of Haar wavelets, i.e.,

$$\langle f_{2^m,0}, f_{2^k,0}\rangle = 0,$$

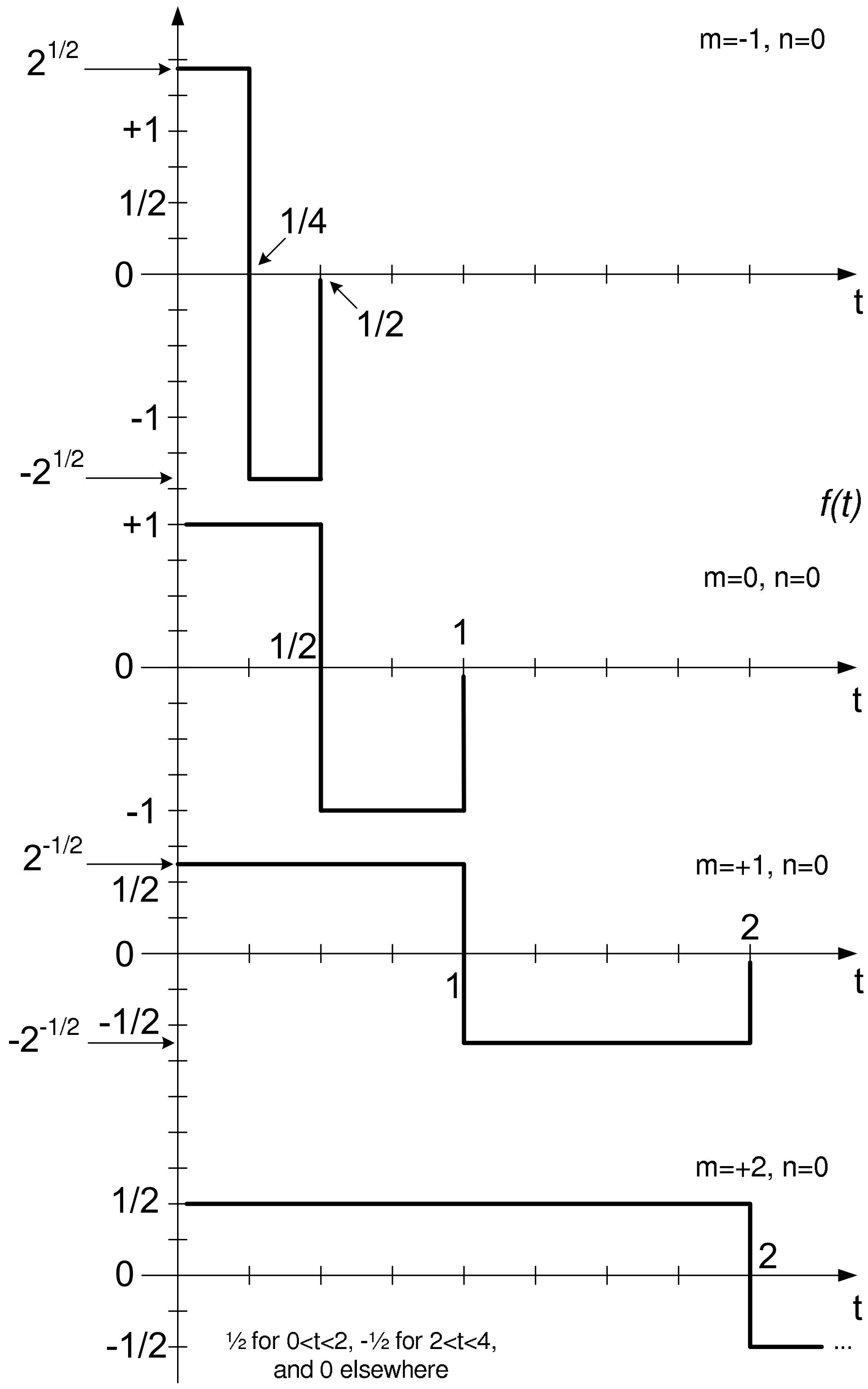
if $k \neq m$. That is, if $m > k$, then $f_{2^m,0}$ is constant over the region of support of $f_{2^k,0}$, and vice versa. Furthermore, we showed in part (c)(i) that each Haar wavelet has unit energy, that is:

$$\langle f_{2^m,0}, f_{2^m,0}\rangle = \mathcal{E}_{f_{2^m,0}} = 1.$$

Therefore, the Haar wavelets (mother and atoms) form a mutually orthonormal set of functions, i.e.,

$$\langle f_{2^m,0}, f_{2^k,0}\rangle = \delta(k-m),$$

where δ denotes the Kronecker delta function.


m=-1, n=0
2^1/2
+1
1/2
1/4
0
1/2
t
-1
-2^1/2
f(t)
+1
m=0, n=0
1/2
1
0
t
-1
2^-1/2
1/2
m=+1, n=0
2
0
1
t
-2^-1/2
-1/2
m=+2, n=0
1/2
0
2
t
-1/2
...
½ for 0<t<2, -½ for 2<t<4,
and 0 elsewhere

LAST Name FOURIER

FIRST Name Jean Baptiste Joseph

Lab Time 365/24/7

Problem	Points	Your Score
Name	10	10
1	20	20
2	40	40
3	30	30
4	40	40
5	40	40
Total	**180**	180