EECS 20. Midterm No. 2, Solution November 19, 2004.

1. **20 points** A system is described by the difference equation

$$y(n) = x(n) + x(n-1) + 0.5y(n-1). \tag{1}$$

   (a) **7 points** Obtain the $[A, b, c^T, d]$ representation of this system by:

      i. choosing the state,

      ii. calculating A, b, c^T, d for your choice of state.

   (b) **8 points** If $x(-1) = 0, y(-1) = 1$, calculate the zero-input (i.e. $x(n) = 0, n \geq 0$) state response.

   Calculate the frequency response H of this system. What are the magnitude and phase of H at $\omega = 0, \pi$?

Answer to 1 (a) (i) Take the state as $s(n) = [x(n-1), y(n-1)]^T$.

(ii) Writing $s(n+1) = As(n) + bx(n)$ in expanded form gives

$$\begin{aligned} s(n+1) &= \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} x(n) \\ x(n) + x(n-1) + 0.5y(n-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(n), \end{aligned}$$

from which

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2}$$

and, since

$$y = [1 \quad 0.5] \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix} + x(n),$$

so $c^T = [1 \quad 0.5], d = 1$.

(b) The zero-input state response is $s(n) = A^n s(0), n \geq 0$. So we need to calculate A^n, with A given in (2). By induction,

$$A^n = \begin{bmatrix} 0 & 0 \\ (0.5)^{n-1} & (0.5)^n \end{bmatrix}$$

and since $s(0) = [0 \quad 1]^T$, $s(n) = [0 \quad (0.5)^n]^T$.

(c) To obtain the frequency response, substitute $x(n) = e^{j\omega n}, y(n) = H(\omega)e^{i\omega n}$ in (1) and simplify to get

$$\forall \omega, \quad H(\omega) = \frac{1 + e^{-i\omega}}{1 - 0.5e^{-i\omega}}.$$

$H(0) = 4, H(\pi) = 0, \angle H(0) = \angle H(\pi) = 0.$

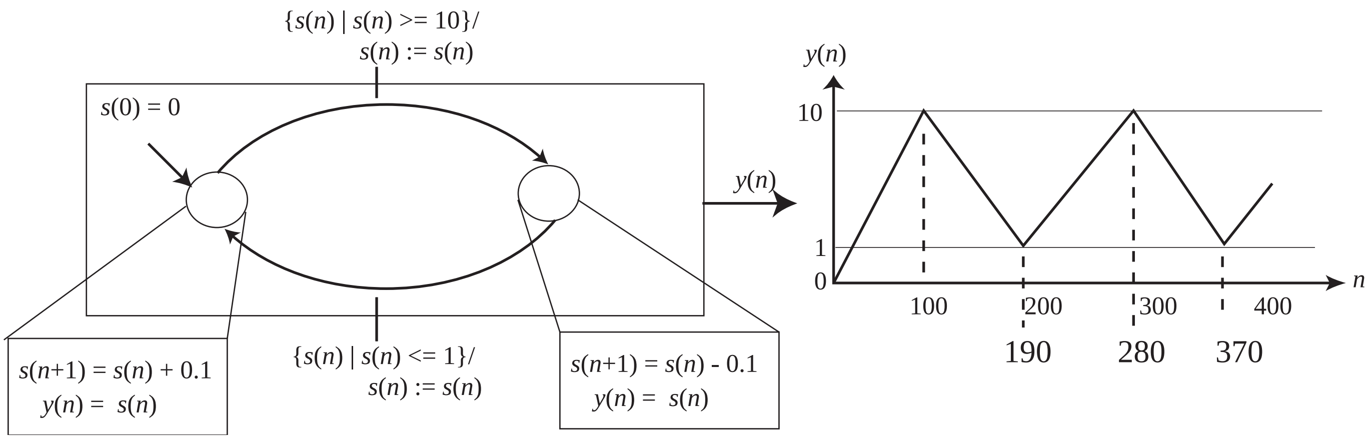


Figure 1: Hybrid system for problem 2

2. **10 points**

   Consider the hybrid system of fig 1.

   (a) **5 points** For $0 \leq n \leq 400$, write down an expression for $s(n)$ and then sketch the output $y(n)$. Carefully mark the times when y reaches its maximum and minimum values.

   (b) **5 points** Does y become periodic after some finite time? If yes, what is the period?

Answer to 2 (a) The expression for s is

$$s(n) = \begin{cases} 0.1n, & 0 \leq n \leq 100 \\ 10 - 0.1(n-100), & 100 \leq n \leq 190 \\ 1 + 0.1(n-190), & 190 \leq n \leq 280 \\ 10 - 0.1(n-280), & 280 \leq n \leq 370 \\ 1 + 0.1(n-370), & 370 \leq n \leq 400 \end{cases}$$

The plot of $y(n) = s(n)$ is shown.

(b) Yes, after $n = 100$, y is periodic and its period is 180 samples.

3. **20 points, 5 points each part** For each of the following discrete-time systems S, state whether it is time-invariant (TI), linear (L), causal (C) and memoryless (ML). For each system prove your answer for the property indicated in **bold**.

   (a) $\forall x, n, \quad S(x)(n) = x(n-1) + x(n)$. **Causal**.

   (b) $\forall x, n, \quad S(x)(n) = x(2n)$. **Time-invariant**.

   (c) $\forall x, n, \quad S(x)(n) = [x(n-1)]^2$. **Memoryless**.

   (d) $\forall x, n, \quad S(x)(n) = 2nx(n-1)$. **Linear**.

Answer to 3
(a) System is TI, L, C and not ML.
Causal Suppose x, u are inputs with outputs y, v respectively. Suppose $x(m) = u(m), m \leq n$. Then $y(n) = x(n-1) + x(n) = u(n-1) + u(n) = v(n)$. So $S(x)(n) = S(u)(n)$.

(b) System is not TI, L, not causal, not ML.
not TI We have

$$\begin{aligned} S \circ D_T(x)(n) &= D_T(x)(2n) = x(2n - T) \\ D_T \circ S(x)(n) &= S(x)(n - T) = x(2(n - T)) \end{aligned}$$

So it is enough to choose x, n, T with $x(2n - T) \neq x(2(n - T))$.

(c) System is TI, not L, causal, not ML.
not ML Suppose, by contradiction, that S is memoryless. So

$$\exists f : R \to R, \forall x, \forall n, S(x)(n) = [x(n-1)]^2 = f(x(n)). \tag{3}$$

In particular, taking $n = 0$ and $x(0) = 0$, we have for all values $x(-1), [x(-1)]^2 = f(0)$. But this is impossible.

(d) S is not TI, L, C, not ML.
Linear Suppose $S(x) = y, S(u) = v$. Then

$$S(ax + bu)(n) = 2n(ax + bu)(n-1) = a2nx(n-1) + b2nu(n-1) = aS(x)(n) + bS(u)(n),$$

so S is linear.

4. **15 points, 5 points each part** A continuous-time LTI system has frequency response

$$\forall \omega, H(\omega) = [1 + i\omega]^{-1}.$$

(a) Plot its magnitude and phase response. Mark the values for $\omega = 0, 1$

(b) The frequency response of another LTI system is

$$\forall \omega, G(\omega) = [H(\omega)]^2.$$

Plot its magnitude and phase response. Mark the values for $\omega = 0, 1$

(c) The signal $\forall t, x(t) = \cos(t) + \cos(10t) + \cos(100t)$ is input to G. Calculate the output, making the approximation $1 + i\omega \approx i\omega$ for $|\omega| > 9$.

Answer to 4

(a) We have

$$|H(\omega)| = \frac{1}{[1+\omega^2]^{1/2}}, \quad \angle H(\omega) = -\tan^{-1}(\omega),$$

so $|H(0)| = 1, \angle H(0) = 0; |H(1)| = 1/\sqrt{2}, \angle H(1) = -\pi/4.$

(b) We have

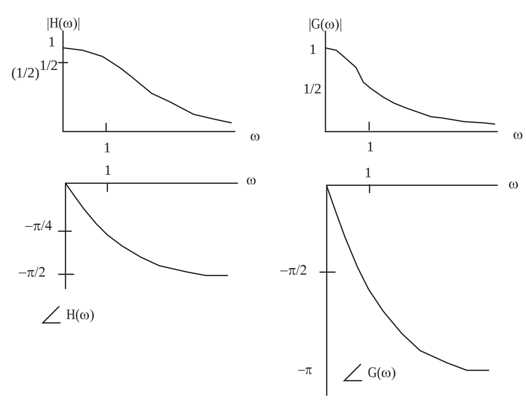
$$|G(\omega)| = |H(\omega)|^2 = \frac{1}{1+\omega^2}; \angle G(\omega) = 2\angle H(\omega) = -2\tan^{-1}(\omega),$$

so $|G(0)| = 1, \angle G(0) = 0; |G(1)| = 1/2, \angle G(1) = -\pi/2.$

(c) The output y is $\forall t$,

$$\begin{aligned} y(t) &= |G(1)|\cos(t + \angle G(1)) + |G(10)|\cos(10t + \angle G(10)) + |G(100)|\cos(100t + \angle G(100) \\ &= \frac{1}{2}\cos(t + \frac{\pi}{2}) + \frac{1}{100}\cos(10t + \pi) + \frac{1}{10000}\cos(100t + \pi). \end{aligned}$$

The responses are plotted in the figure.



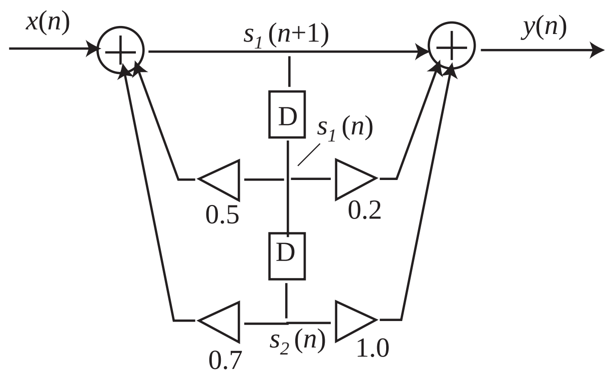


Figure 2: System for problem 5

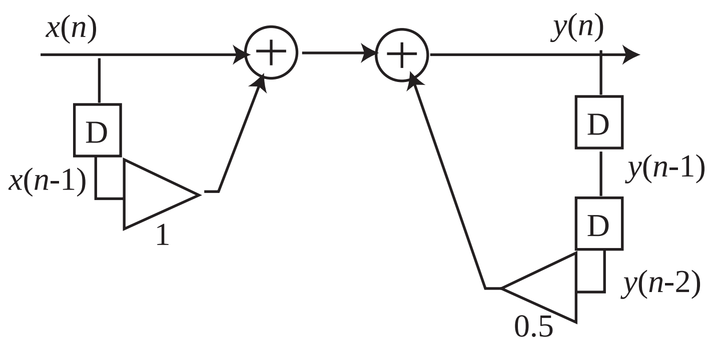


Figure 3: Solution to 5 (a)

5. **15 points** You are given three kinds of building blocks for discrete-time systems: one-unit delay; gains; and adders.

   (a) **5 points** Use these building blocks to implement the system:

   $$y(n) = 0.5y(n-2) + x(n) + x(n-1).$$

   (b) **10 points** For the system of figure 2 obtain its $[A, b, c^T, d]$ representation taking the two-dimensional state at time n to be $s(n) = [s_1(n), s_2(n)]^T$, the output of the two delays.

Answer to 5 (a) The system is given in figure 3.

(b) We have

$$\begin{aligned} \begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} &= \begin{bmatrix} 0.5 & 0.7 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(n) \\ y(n) &= \begin{bmatrix} 0.7 & 1.7 \end{bmatrix} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + [1]\, x(n) \end{aligned}$$

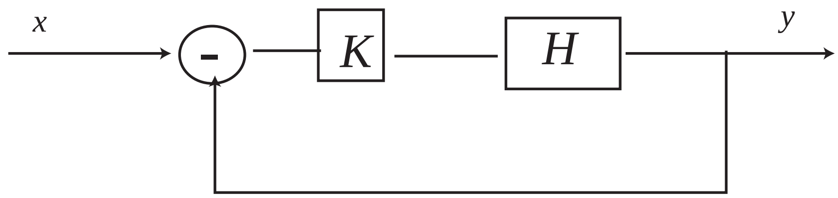


Figure 4: Feedback system for problem 6

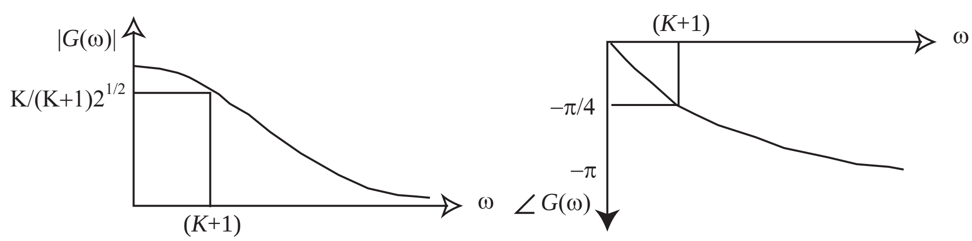


Figure 5: Frequency response for problem 6

6. **20 points** In the negative feedback system of figure 4 assume that $H(\omega) = [1 + i\omega]^{-1}$. Let G_K be the closed-loop frequency response. For $K = 1, 10, 100$

   (a) **10 points** Determine $G_K(\omega), |G_K(\omega)|, \angle G_K(\omega)$. Draw two plots: one for all the magnitude responses $|G_K(\omega)|$, and another for all the phase responses $\angle G_K(\omega)$.

   (b) **5 points** Determine ω_K at which $\angle G_K(\omega_K) = -\pi/4$.

   (c) **5 points** Determine the response y_K of G_K to the input signal $\forall t, x_K(t) = \cos(\omega_K t)$

Answer to 6 The closed loop frequency response is

$$\forall \omega, \quad G(\omega) = \frac{KH(\omega)}{1 + KH(\omega)} = \frac{K}{(K+1) + i\omega}.$$

(a) So

$$|G(\omega)| = \frac{K}{[(K+1)^2 + \omega^2]^{1/2}}, \quad \angle G(\omega) = -\tan^{-1}\frac{\omega}{K+1}.$$

See figure 5

(b) If $\omega_K = K + 1, \angle G_K(\omega_K) = -\tan^{-1} = -\pi/4$.

(c) For all t,

$$y_K(t) = \frac{K}{\sqrt{2}(K+1)}\cos(\omega_K t - \frac{\pi}{4})$$