EECS20n, Solution to Midterm 2, 11/17/00

1. **10 points** Write the following in Cartesian coordinates (i.e. in the form $x + jy$)

   (a) **2 points** $j^3 - j^2 + j + 1 = -j + 1 + j + 1 = 2$

   (b) **2 points** $(1 - j1)/(1 + j1) = -j$

   (c) **2 points** $\sqrt{\cos \pi/4 + j \sin \pi/4} = \pm(\cos \pi/8 + j \sin \pi/8)$

   Write the following in polar coordinates (i.e. in the form $re^{j\theta}$)

   (a) **2 points** $1 + j1 = \sqrt{2}e^{j\pi/4}$

   (b) **2 points** $(1 + j1)/(1 - j1) = e^{j\pi/2}$

2. **10 points** Which of the following discrete-time or continuous-time signals is periodic. Answer yes or no. If the signal is periodic, give its fundamental period and state the units. Suppose that for a discrete-time signal, n denotes **seconds**, and for a continuous-time signal, t denotes **minutes**.

   (a) **2 points** $\forall n \in$ *Ints*, $\quad x(n) = e^{\sqrt{2}n}$ Periodic NO;

   (b) **2 points** $\forall t \in$ *Reals*, $\quad x(t) = e^{\sqrt{2}t}$ Periodic YES; Period = $2\pi/\sqrt{2}$ min

   (c) **2 points** $\forall n \in$ *Ints*, $\quad x(n) = \cos 3\pi n + \sin(3\pi n + \pi/7)$ Periodic YES; Period = 4 sec

   4 points Find A, θ, ω in the following expression:

$$\begin{aligned} A\cos(\omega t + \theta) &= \cos(2\pi \times 10,000t + \frac{\pi}{4}) + \sin(2\pi \times 10,000t + \frac{\pi}{4}) \\ &= \sqrt{2}\cos(2\pi \times 10000t) \end{aligned}$$

   So $A = \sqrt{2}, \omega = 20000, \theta = 0$.

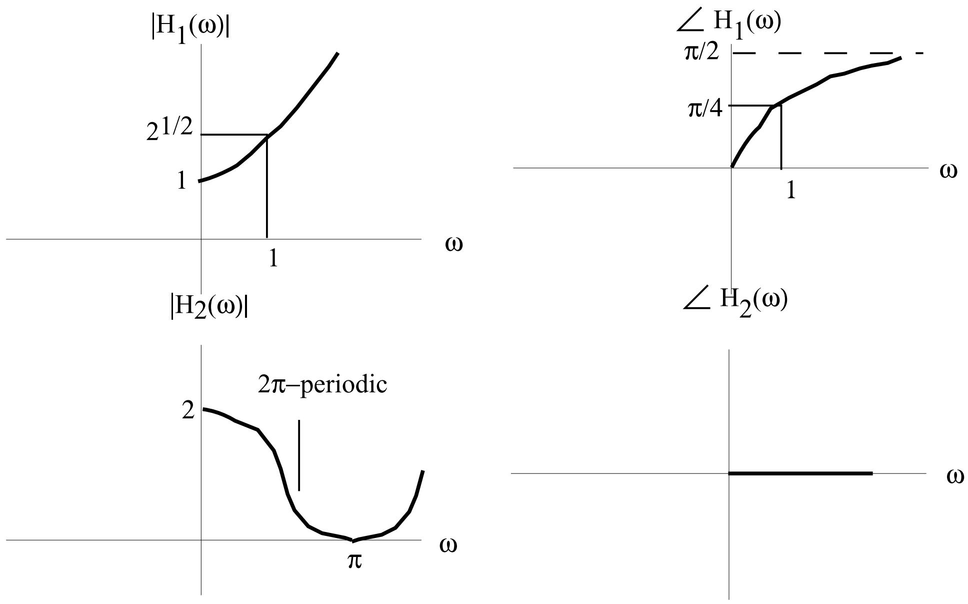

Figure 1: Plots for Problem 3

3. **10 points** On Figure 1 plot the amplitude and phase response of the following frequency responses. On your plots carefully mark the values for $\omega = 0$ and for one other non-zero value of ω.

   (a) **4 points** $\forall \omega \in$ *Reals*, $\quad H_1(\omega) = 1 + j\omega$

   (b) **4 points** $\forall \omega \in$ *Reals*, $\quad H_2(\omega) = 1 + \cos\omega$

   The frequency response is plotted only for $\omega > 0$ since $|H(\omega)|$ is even and $\angle H(\omega)$ is odd.

   2 points Which of H_1, H_2 can be the frequency response of a discrete-time system?

   (a) $|H_1(\omega)| = [1 + \omega^2]^{1/2}$, $\angle H_1(\omega) = \tan^{-1}(\omega)$

   (b) $|H_2(\omega)| = 1 + \cos\omega$, $\angle H_2(\omega) = 0$

   H_2 can be the frequency response of a discrete time system since it is periodic with period 2π.

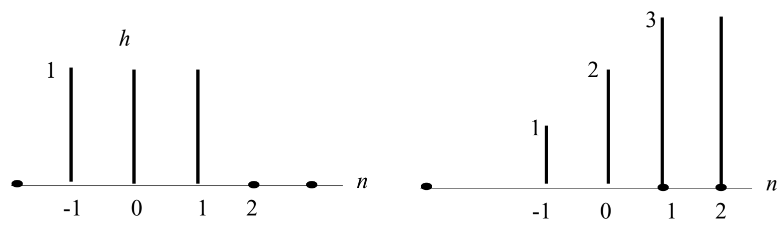

Figure 2: Impulse and step response for Problem 4

4. **10 points** A discrete-time system H has impulse response h : *Ints* $\to$ *Reals* given by

$$h(n) = \begin{cases} 1, & n = -1, 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) **3 points** What is the step response of H, i.e. the output signal when the input signal is *step*, where *step*$(n) = 1, n \geq 0$, and *step*$(n) = 0, n < 0$? You can give your answer as a plot or as an expression.

(b) **3 points** What is the frequency response of H?

(c) **4 points** What is the output signal of H for the following input signals?

i. $\forall n, \quad x(n) = \cos n$

ii. $\forall n, \quad x(n) = \cos(n + \pi/6)$

(a) The step response is

$$\begin{aligned} \forall n, \quad y(n) &= h(-1)\textit{step}(n+1) + h(0)\textit{step}(n) + h(1)\textit{step}(n-1) \\ &= \begin{cases} 0, & n \leq -2 \\ 1, & n = -1 \\ 2, & n = 0 \\ 3, & n \geq 1 \end{cases} \end{aligned}$$

This is also shown on the right in Figure 2

(b) The frequency response is

$$\forall \omega, \quad \hat{H}(\omega) = h(-1)e^{j\omega} + h(0) + h(1)e^{-j\omega} = 1 + 2\cos\omega$$

(c) The response y is

i. $\forall n, \quad y(n) = [1 + 2\cos 1]\cos n$

ii. $\forall n, \quad y(n) = [1 + 2\cos 1]\cos(n + \pi/6)$

5. **10 points**

   (a) **4 points** Find the frequency response for the LTI systems described by these differential equations (input is x, output is y)

      i. $\dot{y}(t) + 0.5y(t) = x(t)$

      ii. $\ddot{y}(t) + 0.5\dot{y}(t) + 0.25y(t) = \dot{x}(t) + x(t)$

   (b) **2 points** What is the response of the first system above for the input $\forall t, \quad x(t) = e^{j(100t+\pi/4)}$?

   (c) **4 points** Find the frequency response for the LTI systems described by these difference equations (input is x, output is y)

      i. $y(n) + 0.5y(n-1) = x(n)$

      ii. $y(n) + y(n-1) + 0.25y(n-2) = x(n) + x(n-1)$

(a) The frequency response is

i. $\hat{H}(\omega) = \frac{1}{j\omega+0.5}$

ii. $\frac{j\omega+1}{-\omega^2+0.5j\omega+0.25}$

(b) The response is

$$\begin{aligned} \forall t, y(t) &= \hat{H}(100)e^{j(100t+\pi/4)} \\ &= \frac{1}{0.5+j100}e^{j(100t+\pi/4)} \approx \frac{1}{100}e^{j(100t-\pi/4)} \end{aligned}$$

(c) (i) $\frac{1}{1+0.5e^{-j\omega}}$

(ii) $\frac{1+e^{-j\omega}}{1+e^{-j\omega}+0.25e^{-2j\omega}}$

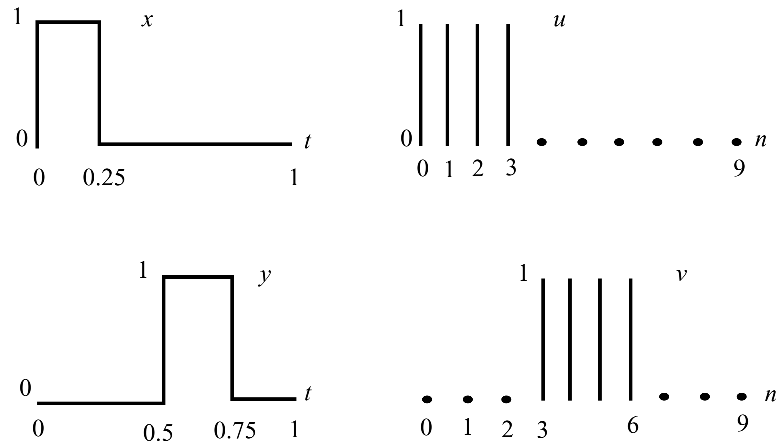

Figure 3: Periodic signals for Problem 6

6. **10 points** Figure 3 plots two continuous-time periodic signals x and y both with period 1 second, and two discrete-time signals u and v both with period 10 samples. The plots are given only for one period. Suppose the exponential Fouriers Series representations of these signals are given as:

$$\forall t \in \textit{Reals}, \quad x(t) = \; = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_x t}$$

$$\forall t \in \textit{Reals}, \quad y(t) = \; = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega_y t}$$

$$\forall n \in \textit{Ints}, \quad u(n) = \sum_{k=0}^{9} U_k e^{jk\omega_u n}$$

$$\forall n \in \textit{Ints}, \quad v(n) = \sum_{k=0}^{9} V_k e^{jk\omega_v n}$$

(a) **2 points** Give the values of $\omega_x = 2\pi$ rad/sec, $\omega_y = 2\pi$ rad/sec, $\omega_u = \pi/5$ rad/sample, $\omega_v = \pi/5$ rad/sample.

(b) **2 points** Calculate the values of the coefficients $X_0 = 0.25$, $Y_0 = 0.25$, $U_0 = 0.4$, $V_0 = 0.4$. These are just the average values ofthe signal over one period.

(c) **3 points** Express y as a delayed version of x and v as a delayed version of u.
$\forall t, y(t) = x(t - 0.5)$, $\forall n, v(n) = x(n - 3)$.

(d) **3 points** Express the FS coefficients $\{Y_k\}$ in terms of $\{X_k\}$ and $\{V_k\}$ in terms of $\{U_k\}$.
$\forall k, \quad Y_k = X_k e^{-jk\pi}, \quad V_k = U_k e^{-jk3\pi/5}$.

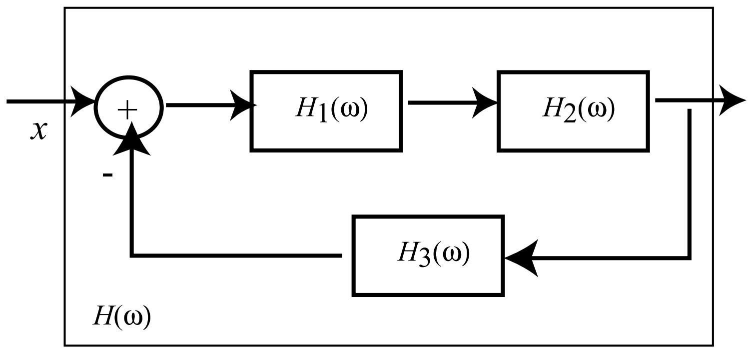

Figure 4: Feedback systems for Problem 7

7. **10 points** Figure 4 shows a feedback system obtained by composing three LTI systems. In the figure, $H_k(\omega), k = 1, 2, 3$ is the frequency response of the three LTI systems.

   (a) **5 points** Calculate the frequency response $H(\omega)$ of the feedback system in terms of the H_k.

   (b) **5 points** Suppose $H_k(\omega) = 1/(1 + j2\omega)$ for all $k = 1, 2, 3$. Calculate $H(0)$, $H(1)$ and $\lim_{\omega\to\infty} H(\omega)$.

   (a) The frequency response is

$$H(\omega) = \frac{H_1(\omega)H_2(\omega)}{1 + H_1(\omega)H_2(\omega)H_3(\omega)} \tag{1}$$

   (b) We have, $H_k(0) = 1, H_k(1) = 1/(1 + 2j), \lim_{\omega\to\infty} H_k(\omega) = 0$. Substituting into (1) gives

$$H(0) = \frac{1}{2}, H(1) = \frac{1 + 2j\omega}{1 + (1 + 2j\omega)^3}, \lim_{\omega\to\infty} H(\omega) = 0$$

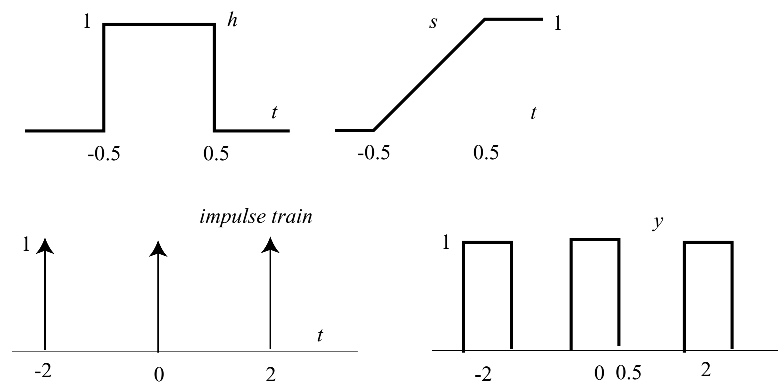

Figure 5: Impulse reponse, step response, impulsetrain and response for Problem8

8. **10 points** A continuous-time LTI system has the impulse response

$$\forall t \in \textit{Reals}, \quad h(t) = \begin{cases} 1, & |t| < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

(a) **2 points** Sketch the impulse response, and mark carefully the relevant points on your plot.

(b) **2 points** Is this system causal? Answer yes or no.

(c) **2 points** Sketch the step response of this system, i.e. the response to the input signal *step*$(t) = 1, t \geq 0$ and $= 0, t < 0$?

(d) **2 points** Consider the input signal *impulsetrain*, where

$$\forall t \in \textit{Reals}, \quad \textit{impulsetrain}(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k).$$

Sketch *impulsetrain*.

(e) **2 points** Sketch the response of the system to *impulsetrain*.

(a) The impulse response h is shown in Figure 5

(b) The system is NOT causal.

(c) The step response s is the integral of the impulse response as shown.

(d) The impulse train is as shown.

(e) Its response is

$$\forall t, y(t) = (h * \textit{impulsetrain})(t) = \sum_{k=-\infty}^{\infty} h(t - 2k)$$