EE 127 / EE 227AT

Midterm 1 Spring 2016

Please write your answers on these sheets, use the back sides if needed. Show your work. You can use a fact from the slides/book without having to prove it unless you are specifically asked to do so. Be organized and use readable handwriting. There is a page for scratch work at the end.

Exercise 1 (Solution of optimization problems.) Give specific examples of functions $f_0 : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ such that the optimization problem $\min_x f_0(x)$ subject to $f(x) \leq 0$ has the following properties. Only give one example per case for a total of three examples. Please no drawings. Give the formulae for f_0 and f.

(a) (5 pts.) The set of optimal solutions contains one point.

(b) (5 pts.) The set of optimal solutions contains an infinite number of points.

(c) (5 pts.) The set of optimal solutions is empty and there is a constant $a \in \mathbb{R}$ such that $f_0(x) \geq a$ for all $x \in \mathbb{R}^n$.

Exercise 2 (Matrix norms.) (15 pts.) A matrix $A \in \mathbb{R}^{m,n}$ with rank r has singular values $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. Prove that the spectral norm satisfies $\|A\|_2^2 = \sigma_1^2$.

Exercise 3 (Matrix approximation.) For a given $A \in \mathbb{R}^{m,n}$, with $\text{rank}(A) = r$, consider the problem

$$\min_{A_k \in \mathbb{R}^{m,n}} \|A - A_k\|_F^2 \text{ subject to } \text{rank}(A_k) = k.$$

Let $\Sigma_{i=1}^r \sigma_i u_i v_i^\top$ be a singular value decomposition of A. For $k \leq r$, it is known that an optimal solution of the problem is $A_k = \Sigma_{i=1}^k \sigma_i u_i v_i^\top$.

(a) (5 pts.) Suppose that $r = 4$ and $\sigma_1 = 4$, $\sigma_2 = 2$, $\sigma_3 = 2$, and $\sigma_4 = 1$. Quantify the relative error in A_k compared to the "true" matrix A for $k = 1, 2, 3,$.

(b) (10 pts.) Suppose $m \geq n$ and $\text{rank}(A) = n$. Formulate an optimization problem that determines how "far" A is from being of rank $n - 1$. Solve this problem and obtain an explicit expression for a matrix $B \in \mathbb{R}^{m,n}$ such that $A + B$ has rank $n - 1$. (Ignore what was given in part a.)

Exercise 4 (Optimization over norm balls.) (10 pts.) For a given $y \in \mathbb{R}^n$, derive an optimal solution of the problem $\max x^\top y$ subject to $\|x\|_\infty \leq 1$.

Exercise 5 (Projection on a hyperplane.) Consider the hyperplane $\{z \in \mathbb{R}^n \ : \ a^\top z = b\}$, $a \neq 0$, and a point $y \in \mathbb{R}^n$.

(a) (10 pts.) Determine the Euclidean projection of y onto the hyperplane.

(b) (5 pts.) Determine the Euclidean distance between y and its projection on the hyperplane.

Exercise 6 (Properties of dyad.) Let $x, y \in \mathbb{R}^n$, both not identical to the zero vector, and $A = xy^\top \in \mathbb{R}^{n,n}$.

(a) (5 pts.) Determine an eigenvalue and an eigenvector of A.

(b) (5 pts.) We know that A has rank one. Write a proof of this fact.

(c) (5 pts.) What is the dimension of $\mathcal{N}(A)$?

(d) (5 pts.) Compute a singular value decomposition of A and write it in compact form.

Exercise 7 (Bound on a polynomial's derivative.) (10 pts.) For $w \in \mathbb{R}^{k+1}$, we define the polynomial p_w, with values

$$p_w(x) \doteq w_1 + w_2 x + \ldots + w_{k+1} x^k.$$

Prove that

$$\forall\, x \in [-1, 1] \;:\; \left| \frac{\mathrm{d}p_w(x)}{\mathrm{d}x} \right| \leq k^{3/2} \|v\|_2,$$

where $v = (w_2, \ldots, w_{k+1}) \in \mathbb{R}^k$.

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