EE126: Probability and Random Processes **SP'10**

Midterm — April 1

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Formulas: Given the short attention span induced by twitter and the like, we thought you might appreciate not having to remember the following formulas. After all, they are on Wikipedia.

$$\mathbf{X} = N(\mu, \Sigma) \Leftrightarrow f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\}$$
$$L[\mathbf{X}|\mathbf{Y}] = E(\mathbf{X}) + \Sigma_{\mathbf{XY}}\Sigma_{\mathbf{Y}}^{-1}(\mathbf{Y} - E(\mathbf{Y}))$$
$$\text{cov}(\mathbf{AX}, \mathbf{BY}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}^T.$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$P(V > 1.64) = 0.05 \text{ when } V = N(0,1).$$

Problem 1. *(***Multiple Choice 20%***)*

- *If X, Y, Z are* pairwise *independent, then XY and Z are (circle the correct answer(s)): uncorrelated, independent, possibly neither.*

- *Let X, Y, Z be general random variables. Then (circle the identities that always hold):*

$$E[L[X|Y]|Y] = E[X|Y]$$
$$E[L[X|Y]|Y] = L[X|Y]$$
$$L[L[X|Y,Z]|Y] = L[X|Y]$$
$$L[XY|Y] = YL[X|Y].$$

- *Jointly Gaussian random variables that are pairwise independent are also mutually independent (circle the correct answer):*
  True, False.

- *The maximum of two independent exponentially distributed random variables is exponentially distributed (circle the correct answer):*
  True, False.

- *Let X, Y be jointly Gaussian, zero mean, and unit variance random variables. Then (circle the statements that are certainly true):*

$$cov(X,Y) \le 1;\ cov(X,Y) \le 0.5;\ P(X > Y) = 1/2;\ L[X|Y] = E[X|Y].$$

Problem 2. *(***Quick Calculations 20%***) Let* X, Y, Z *be i.i.d.* $U[0,1]$. *Calculate*

(a) $E[(X+Y)^2|X]$

(b) $E[(X+Y)(Y+Z)|Y]$

(c) $L[X+Y|X+Y+Z]$

(d) $L[(X+Y)Y|Y]$

(e) $E[\cos(X+Y)|Y]$

Problem 3. *(***More Quick Calculations 20%***) Let X, Y, Z be i.i.d. $N(0,1)$. Calculate:*
[Hint: First calculate $\text{cov}(X+Y, X-Y)$.]

(b) $E[2X + Y | X - Y]$

(c) $E[(X + Y + Z)^2 | X - Y]$

(d) $E[X + 2Y + 3Z | X + Z, Y + 2Z]$

Problem 4. *(**20%**)*

(a) Write the projection characterization of $E[X|Y]$*.*

(b) Use this property to show that if Z *is independent of* (X,Y)*, then* $E[Xg(Y)Z|Y] = E[X|Y]g(Y)E(Z)$*.*

Problem 5. *(**20%**) When* $X = 0, \mathbf{Y} = N(0, \Sigma)$ *and when* $X = 1, \mathbf{Y} = N(\mu, \Sigma)$ *where*

$$\mu = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}.$$

(a) Find $MLE[X|\mathbf{Y}]$ *and express it in the form* $MLE[X|\mathbf{Y}] = 1\{\mathbf{a}^T\mathbf{Y} \geq \alpha$.

(b) Find $\hat{X}$ *based on* Y *and taking values in* $\{0, 1\}$ *that maximizes* $P[\hat{X} = 1|X = 1]$ *subject to* $P[\hat{X} = 1|X = 0] \leq 0.05$.