EE126: Probability and Random Processes **SP'10**

Midterm — February 18

Lecturer: Jean C. Walrand SOLUTIONS *GSI: Baosen Zhang*

Problem 1. (**Multiple Choice 20%**)

- *Assume* $P(A) = 0.2, P(B) = 0.6, P(A \cup B) = 0.5$. *Then* $P[A|B] = ...$
  (Circle one): 0, 0.1, 0.2, 0.3, 0.4, (0.5), 0.6, 1.
  $P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cap B) = 0.3 \Rightarrow P[A|B] = 0.3/0.6 = 0.5.$

- *Assume that* $X = n$ *with probability* $\alpha 3^{-n}$ *for* $n \geq 1$. *Then* $E(X) = ...$
  (Circle one): 0.25, 0.5, 1, (1.5), 2, 2.5, 3, 4, 5, 6, depends on α.
  Since $\sum_{n=1}^{\infty} 3^{-n} = 1/2, \alpha = 2$. *Then* $E(X) = 2\sum n3^{-n} = 2a(1-a)^{-2}$ *with* $a = 1/3$. *So* $E(X) = 1.5$.

- *Assume that* $\Omega = [0,1], \mathcal{F} = \{\emptyset, [0,0.3], (0.3,1], [0,1]\}$ *and* $P([0,0.3]) = 0.4$. *Which of the following functions of* ω *are random variables? (Circle the correct answers):*
  $X(\omega) = 1\{\omega \leq 0.4\}$, $(X(\omega) = 1)$, $(X(\omega) = 2 + 3 \times 1\{\omega \leq 0.3\})$, $X(\omega) = 1\{\omega \leq 0.5\}$.
  For X *to be a RV, we need* $\{\omega \mid X(\omega) \leq x\} \in \mathcal{F}, \forall x$.

- *Assume that* $X \geq 0, E(X) = 4$, *and* $\text{var}(X) = 0.3$. *Circle the statements that are certainly true:*
  $(P(X > 10) \leq 0.4)$, $(P(X > 10) \leq 0.5)$, $(P(X > 10) \leq 0.6)$, $P(|X - 4| > 1) \leq 0.2$, $(P(|X - 4| > 2) \leq 0.075)$, $P(|X - 1| > 3) \leq 0.2$.
  We use Chebyshev's and Markov's inequalities.

- *Assume that* X *is exponentially distributed with rate 2. Then (circle the true statements):*
  $E(X) = 2$, $(E(X) = 0.5)$, $(P[X > 2|X > 1] > P(X > 2))$,
  $P[X > 1|X > 2] < P(X > 1)$, $(P[X > 3|X > 1] = P(X > 2))$.
  We use the memoryless property and some obvious facts.

- *Assume that* X *is uniformly distributed in* $[0,1]$. *Then (circle the correct statements):*
  $(E(\cos(X)) = \sin(1))$, $E(X^n) = 2n$, $(E(X^n) = 1/(n+1))$, $\text{var}(X) = 1/4$, $(\text{var}(X) = 1/12.)$
  This should be obvious. For instance $E(\cos(X)) = \int_0^1 \cos(x)dx = [\sin(x)]_0^1 = \sin(1)$.

- *Let* (X, Y) *be uniformly picked in the triangle with vertices* $(0,0), (0,1), (1,0)$. *Circle the correct statements:*
  X, Y *are independent.* X, Y *are positively correlated.* X, Y *are uncorrelated.* $(X, Y$ *are negatively correlated).*

- *Complete the following sentence:* A random variable is a .... *function of the outcome of a random experiment.*

Problem 2. *(20%) A random number generator of type n selects a number uniformly in the set $\{1, 2, \ldots, n\}$. You are given such a generator and you are told that its type is n with probability $n2^{-n-1}$ for $n \geq 1$. Let X be the value that the random number generator selects. Given X, what is the probability that the type of the generator is n, for $n \geq 1$?*

We use Bayes' rule. We find

$$\begin{aligned} P[\text{ Type is } n \mid X = x] &= \frac{P[X = x \mid \text{ Type is } n]P(\text{ Type is } n)}{\sum_m P[X = x \mid \text{ Type is } m]P(\text{ Type is } m)} \\ &= \frac{(1/n)1\{x \leq n\}n2^{-n-1}}{\sum_m (1/m)1\{x \leq m\}m2^{-m-1}} \\ &= \frac{1\{x \leq n\}2^{-n-1}}{\sum_{m \geq x} 2^{-m-1}} = 2^{x-n-1}1\{x \leq n\}. \end{aligned}$$

In other words,

$$P[\text{ Type is } x + n \mid X = x] = \frac{1}{2^{n+1}}, n = 0, 1, \ldots.$$

Problem 3. *(20%) The point (X, Y) is picked uniformly in the triangle with vertices $(0,0),(0,1),(1,1)$. Calculate $\text{cov}(X, Y)$, the covariance of X and Y. Recall that $\text{cov}(X, Y) = E(XY) - E(X)E(Y)$.*

First, we calculate $E(X)$. We find

$$\begin{aligned} E(X) &= \int\int xf(x,y)dxdy = \int_0^1 \int_0^x x2dxdy \\ &= \int_0^1 2x^2 dx = \frac{2}{3}. \end{aligned}$$

Second, we calculate $E(Y)$. We find

$$\begin{aligned} E(Y) &= \int\int yf(x,y)dxdy = \int_0^1 \int_0^x y2dxdy \\ &= \int_0^1 x^2 dx = \frac{1}{3}. \end{aligned}$$

Third, we calculate $E(XY)$. We find

$$\begin{aligned} E(XY) &= \int\int xyf(x,y)dxdy = \int_0^1 \int_0^x xy2dxdy \\ &= \int_0^1 x^3 dx = \frac{1}{4}. \end{aligned}$$

Finally, we put the pieces together and we get

$$\text{cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{3} \times \frac{1}{3} = \frac{1}{36}.$$

Note that X and Y are positively correlated, as we expected.

Problem 4. *(**20%**) Let X be the random variable selected as follows: with probability 0.2, $X = 0.8$; with probability 0.3, $X = 1.4$; with probability 0.5, X is picked uniformly in $[0, 2]$. (a) Give the pdf $f(x)$ of X; (b) Give the cpdf $F(x)$ of X; (c) Calculate $E(X)$; (d) Calculate var(X).*

(a)

$$f(x) = 0.25 \times 1\{0 \leq x \leq 2\} + 0.2\delta(x - 0.8) + 0.3\delta(x - 1.4).$$

(b)

$$F(x) = \begin{cases} 0, \text{ if } x \leq 0 \\ 0.25x, \text{ if } 0 < x < 0.8 \\ 0.25x + 0.2, \text{ if } 0.8 \leq x < 1.4 \\ 0.25x + 0.5, \text{ if } 1.4 \leq x \leq 2 \\ 1, \text{ if } x \geq 2. \end{cases}$$

(c)

$$E(X) = \int xf(x)dx = \int_0^2 0.25xdx + 0.2 \times 0.8 + 0.3 \times 1.4 = 0.5 + 0.16 + 0.42 = 1.08.$$

(d) We first calculate $E(X^2)$. One finds

$$\begin{aligned} E(X^2) &= \int x^2 f(x)dx = \int_0^2 0.25x^2 dx + 0.2 \times (0.8)^2 + 0.3 \times (1.4)^2 \\ &= 0.25\frac{2^3}{3} + 0.2 \times 0.64 + 0.3 \times 1.96 = 0.667 + 0.128 + 0.588 = 1.383 \end{aligned}$$

Finally, we find

$$\text{var}(X) = E(X^2) - E(X)^2 = 1.383 - (1.08)^2 = 1.383 - 1.166 \approx 0.117.$$

Problem 5. *(10%) Let $F(x)$ be the cpdf of a random variable X. Prove that $F(x)$ is right-continous.*

Let $x_n \downarrow x$. Then $(-\infty, x_n] \downarrow (-\infty, x]$. Consequently,

$$F(x_n) = P((-\infty, x_n]) \downarrow P((-\infty, x]) = F(x).$$

Problem 6. *(10%) For $n \geq 1$, let $X_n(\omega) = 1\{\omega \in A_n\}$ where the events A_n are such that $P(A_n) = 1/(4n^2)$. Show that $X := \sum_{n=1}^{\infty} X_n$ is finite with probability 1.*

Note that $\sum_n P(A_n) < \infty$. Hence, $P(A_n, \text{ i.o.}) = 0$. This means that, with probability one, ω belongs to only finitely many A_n's. Thus, with probability one, only finitely many X_n's are nonzero. Thus, with probability one, X is a sum of only finitely many nonzero terms and is therefore finite.