Name: ____________________________

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UNIVERSITY OF CALIFORNIA

College of Engineering
Department of Electrical Engineering and Computer Sciences

Professor Tse　　　　　　　　　　　　　　　　　　　　　　　　　　　　　Spring 2004

EECS 126 — Midterm #1

- **Closed book, no notes, no calculators.**
- **Please write your solutions on the blank pages (provided) only.**
- **Please show your steps clearly. True/false answer without explanation gets no mark.**
- **All random variables referred to are discrete.**

Some pmf's:

Binomial [Bin(n, p)]: $P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, \ldots, n$

Poisson (λ): $P_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$

Geometric (p): $P_X(x) = (1-p)^{x-1} p, \quad x = 1, 2, \ldots$

Question	**Points possible**	**Your points**
1	5	
2	10	
3	10	
4	15	
5 – Part I	34	
5 – Part II	26	
Total	100	

[5 pts.] **1.** State the total probability rule and Bayes' rule for: (i) events, (ii) discrete random variables.

[10 pts.] 2. Consider n independent flips of a coin and the events

$A = \{3^{\text{rd}}$ flip is a head$\}$
$B = \{5^{\text{th}}$ flip is a head$\}$
$C = \{5^{\text{th}}$ and 7^{th} flip are heads$\}$
$D = \{5^{\text{th}}$ or 7^{th} flip are heads$\}$

[2] a) Find all pairs of events that are disjoint.

[2] b) Find all pairs of events that are independent.

[2] c) Find all pairs of events such that one event contains the other, and specify which contains the other.

[4] d) Draw a Venn diagram showing the relationships between A, B, C, and D.

[10 pts.] 3. True or false? If true, prove the statement. If false, give a counter-example.

X, Y, Z are three random variables.

[5] a) If X and Y are independent, then X and Y are independent conditional on Z.

[5] b) If X and Y are independent conditional on Z, then X and Y are independent.

X and Y are independent conditional on Z if

$$P(X = i, Y = j | Z = k) = P((X = i | Z = k) \cdot P(Y = j | Z = k) \text{ for all } i, j, k).$$

[15 pts.] 4a. Show that

$$Var(X) = E(X^2) - E(X)^2. \quad \textbf{[3 pts.]}$$

[3] b) Compute the mean of a geometric random variable by conditioning on the result of the first trial.

[9] c) Use a similar method to compute the **variance** of a geometric random variable. (Hint: Part (a) may be useful.)

5. In a wireless link, the channel is sometimes in a "deep fade," in which case a packet that is transmitted will be lost. This can be modelled by an **erasure** channel. In an erasure channel, each packet is erased with probability p, in which no information is obtained, or is received intact with probability $(1 - p)$.

To increase the reliability, **coding** is often performed on the packets:

Figure 1

In the example above, one packet gets duplicated into two packets that are sent successively over the wireless link and the first one is erased.

[34 pts.] Part I

To analyze the performance of a coding scheme, one must have a model for what happens to multiple packets, not only to individual packets.

[2 pts.] a. Construct a sample space Ω to describe the scenario where two packets are transmitted.

[2 pts.] b. What is the probability law on Ω to model the case when the two packets are independently erased?

[6 pts.] c. Now suppose the erasures of the two packets are **dependent** and the probability of the second packet getting erased conditional on the first packet erased is q. Is that enough information to specify the probability law on Ω? If so, specify it. If not, specify any additional parameters.

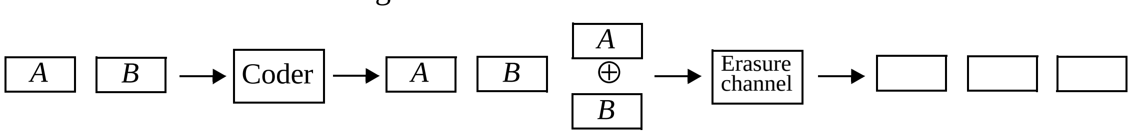
[6 pts.] d. Compute the probability of successfully communicating packet A in the duplication scheme in Fig. 1. Do the calculation for both the model in (b) and (c).

[6 pts.] e. In a wireless link, the "deep fade" can last a considerable amount of time so that loss of successive packets can be highly dependent. Which of the schemes below do you think will do better? Justify your answer using (d), above.

(i)
duplicated packets transmitted back-to-back

(ii)
duplicated packets interleaved so that each pair is far apart in time

[12 pts.] f. Consider a different coding scheme



that takes two packets A and B and transmits three packets A, B, C where $C = A \oplus B$ is obtained by pairwise XORing the bits in A and B $(0 \oplus 0 = 0, 0 \oplus 1 = 1 \oplus 0 = 1, 1 \oplus 1 = 0)$.

Find a natural scheme for decoding A and B given the output of the erasure channel. Assuming the independent erasure model, calculate the probability of successfully decoding packet A. Which scheme is better, this scheme or the duplication scheme? Explain.

[26 pts.] 5 – Part II.

[5 pts.] a. Now suppose there is feedback from the receiver to the transmitter and the transmitter will repeatedly transmit the same packet until the receiver acknowledged that it has received it intact. Calculate the pmf and the expected number of packet transmissions assuming packet erasures are independent.

[15 pts.] b. Now suppose the erasures are dependent such that the probability of erasure of a packet given the past is q if the most recent transmitted packet is erased and $1 - q$ if the most recent packet is not erased. Calculate the pmf and the expected number of transmissions in the feedback scheme in II (a), above. Is the pmf geometric? For which value of q is the expected time **longer** than in the independent case?

[6 pts.] c. Now suppose that the feedback link can also fade and the probability that the acknowledgement is erased is r. Assuming that both the erasures in the forward and feedback link are independent over time and independent of each other, calculate the pmf and expected number of packet transmissions.