SOLUTIONS

Problem 1: You flip a fair coin repeatedly. What is the probability that you have to flip it exactly 10 times to see two “heads”?

Solution:

There must be exactly one head among the first nine flips and the last flip must be another head. The probability of that event is

\[ 9 \left( \frac{1}{2} \right)^9 \times \left( \frac{1}{2} \right) = \frac{9}{2^{10}}. \]
Problem 2: Let $A, B, C$ be three events. Assume that
$P(A) = 0.6$, $P(B) = 0.6$, $P(C) = 0.7$, $P(A \cap B) = 0.3$, $P(A \cap C) = 0.4$, $P(B \cap C) = 0.4$, and $P(A \cup B \cup C) = 1$. Find $P(A \cap B \cap C)$.

Solution:
We know that (draw a picture)

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Substituting the known values, we find

$$1 = 0.6 + 0.6 + 0.7 - 0.3 - 0.4 - 0.4 + P(A \cap B \cap C),$$

so that

$$P(A \cap B \cap C) = 0.2.$$
Problem 3: There are two coins. The first coin is fair. The second coin is such that \( P(H) = 0.6 = 1 - P(T) \). You are given one of the two coins, with equal probabilities between the two coins. You flip the coin four times and three of the four outcomes are \( H \). What is the probability that your coin is the fair one?

Solution:

Let \( A \) designate the event “your coin is fair.” Let also \( B \) designate the event “three of the fair outcomes are \( H \).”

We know that

\[
P[A|B] = \frac{P(A \cap B)}{P(A)} = \frac{P[B|A]P(A)}{P[B|A]P(A) + P[B|A^c]P(A^c)}
\]

\[
= \frac{C(4,3)(1/2)^4}{C(4,3)(1/2)^4 + C(4,3)(0.6)^3(0.4)} = \frac{2^{-4}}{2^{-4} + (0.6)^30.4}.
\]
Problem 4: Define the random variable $X$ as follows. You throw a dart uniformly in a circle with radius 5. The random variable $X$ is equal to 2 minus the distance between the dart and the center of the circle if this distance is less than or equal to one. Otherwise, $X$ is equal to 0.

a. Plot carefully the probability distribution function $F(x) = P(X \leq x)$ for $x \in \mathbb{R} := (-\infty, +\infty)$.

b. Give the mathematical expression for the probability density function $f(x)$ of $X$ for $x \in \mathbb{R} := (-\infty, +\infty)$.

Solution:

Let $Y$ be the distance between the dart and the center of the circle.

a. When $1 \leq x \leq 2$, $X \leq x$ if $Y \geq 2 - x$, which occurs with probability $\frac{(25 - (2-x)^2)}{25}$. Also, $X = 0$ if $Y > 1$, which occurs with probability $\frac{(25 - 1)}{25} = \frac{24}{25}$. These observations translate into the plot shown below:

b. Taking the derivative of $F(x)$, one finds

$$f(x) = \frac{24}{25} \delta(x) + \frac{2x - 4}{25} 1\{1 < x < 2\}.$$