EECS 126 MIDTERM 2 FALL 99 SOLUTIONS.

1.

a) $F_V(v) = P(V \leq v) = \int_0^v \int_0^v f_{X,Y}(x,y)dxdy = \begin{cases} v^2 & 0 \leq v \leq 1 \\ 0 & \text{else} \end{cases} \Rightarrow f_V(v) = \frac{d}{dv}F_V(v) = \begin{cases} 2v & 0 \leq v \leq 1 \\ 0 & \text{else} \end{cases}$

(Check that $\int_{-\infty}^{\infty} f_V(v)dv = 1$!)

Similarly, $f_W(w) = \begin{cases} 2-2w & 0 \leq w \leq 1 \\ 0 & \text{else} \end{cases}$

b) $E[V|V > \frac{1}{2}] = \int_{-\infty}^{\infty} v f_V(v|V > \frac{1}{2})dv$.

With $f_V(v|V > \frac{1}{2}) = \begin{cases} \frac{f_V(v)}{P(V > \frac{1}{2})} = \frac{f_V(v)}{\frac{3}{4}} = \frac{4}{3}f_V(v) & \frac{1}{2} \leq v \leq 1 \\ 0 & \text{else} \end{cases}$, we get $E[V|V > \frac{1}{2}] = \int_{\frac{1}{2}}^{1} v2v\frac{4}{3} = \frac{7}{9}$.

c) $U = V - W = max(X,Y) - min(X,Y) = |X - Y|$

$\Rightarrow F_U(u) = P(U \leq u) = P(|X - Y| < u) =$ similarly to (a) $= 1 - (1-u)^2$

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2. We have $X \sim Exp(\lambda)$.

a) Generate $Y \sim Exp(\mu)$ by applying a function g to X.

We know that if $Y = aX$, then $f_Y(y) = \frac{1}{a}f_X(\frac{y}{a})$. In this case, $f_Y(y) = \mu e^{-\mu y} = \frac{1}{a}f_X(\frac{y}{a}) = \frac{1}{a}\lambda e^{-\lambda\frac{y}{a}}$; from this, we see that $\frac{\lambda}{a} = \mu$, so that $a = \frac{\lambda}{\mu}$.

Thus, $g(X) = \frac{\lambda}{\mu}X$.

b) Generate Y, where $F_Y(y) \sim Uniform(0,1)$.

$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = g$ strictly incr. $= P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) =$ (since Y has to be Uniform(0,1)) $=$ y.
We see that we have to choose $g^{-1}(y) = F_Y^{-1}(y)$, so that $g(X) = F_X(X)$.

3.

a) S =aggregate incoming traffic rate at time 0, $S = \sum_{i=1}^{n} X_i \Rightarrow S \sim N(n\mu, n\sigma^2)$

We need $P(S > c) = 1 - P(S \leq c) = 1 - \Phi\left(\frac{c-n\mu}{\sqrt{n}\sigma}\right) = 10^{-3}$.

$\Rightarrow \Phi\left(\frac{c-n\mu}{\sqrt{n}\sigma}\right) = 0.999$

$\Rightarrow \frac{c-n\mu}{\sqrt{n}\sigma} = \Phi^{-1}(0.999)$

$\Rightarrow n + \sqrt{n}\frac{\sigma}{\mu}\Phi^{-1}(0.999) - \frac{c}{\mu} = 0$

$\Rightarrow \sqrt{n} = -\frac{\sigma}{2\mu}\Phi^{-1}(0.999) + \sqrt{\frac{\sigma^2}{4\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu}}$

$\Rightarrow n = \frac{\sigma^2}{2\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu} + \frac{\sigma}{\mu}\Phi^{-1}(0.999)\sqrt{\frac{\sigma^2}{4\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu}}$

$\Rightarrow$ We can at most accomodate $\frac{\sigma^2}{2\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu} + \frac{\sigma}{\mu}\Phi^{-1}(0.999)\sqrt{\frac{\sigma^2}{4\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu}}$ users.

b) $E[X_i] = E[Z] + E[Y_i] = \mu$, $Var(X_i) = Var(Z) + Var(Y_i) = \sigma^2$

$$\begin{aligned}
Cov(X_i, X_j) &= E[X_i X_j] - E[X_i]E[X_j] \\
&= E[Z^2 + ZY_i + ZY_j + Y_iY_j] - (E[Z] + E[Y_i])(E[Z] + E[Y_j]) \\
&= E[Z^2] + E[ZY_i] + E[ZY_j] + E[Y_iY_j] - E[Z^2] - E[Z]E[Y_i] - E[Z]E[Y_j] - E[Y_i]E[Y_j] \\
&= Z, Y_i \text{ are all independent} \\
&= E[Z^2] - E[Z]^2 \\
&= Var(Z) = \frac{\sigma^2}{2}
\end{aligned}$$

c) Still, $S = \sum_{i=1}^{n} X_i$.

$E[S] = nE[X_i] = n\mu$

$Var(S) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} Cov(X_i, X_j) = nVar(X_i) + n(n-1)Cov(X_i, X_j) = n\sigma^2 + n(n-1)\frac{\sigma^2}{2}$
$\Rightarrow S \sim N(n\mu, n\sigma^2(1 + \frac{n-1}{2}))$.

Similarly to (a), we get

$$n \leq \frac{(1+\frac{n-1}{2})\sigma^2}{2\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu} + \frac{(1+\frac{n-1}{2})\sigma}{\mu}\Phi^{-1}(0.999)\sqrt{\frac{(1+\frac{n-1}{2})\sigma^2}{4\mu^2}(\Phi^{-1}(0.999))^2 + \frac{c}{\mu}}.$$

This quantity is smaller than the one on (a). Intuitively: In (a), peaks will, on average, “cancel out” - which is not the case here.

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4. N = number of packets arriving in [0,1]; A = number of packets routet to A in [0,1]

a)$E[A] = E[E[A|N]] = E[pE[N]] = pE[N] = p\lambda$

b) $X_i = \begin{cases} 1 & \text{packet routet to A (with probability p)} \\ \\ 0 & \text{packet routet to B (with probability (1-p)} \end{cases}$

$M_A(s) = M_N(S)|_{e^s = M_{X_i}(s)} = e^{\lambda(1-p+pe^s-1)} = e^{\lambda p(e^s-1)}$; from this, we see that A is Poisson with parameter $p\lambda$.

Alternative solution: $P(A=a) = \sum_{k=0}^{\infty} P(A=a|N=k)P(N=k)$, where $P(A=a|N=k) = \begin{cases} \binom{k}{a} p^a (1-p)^{k-a} & a \leq k \\ \\ 0 & \text{else} \end{cases}$

$$\begin{aligned}
\text{Thus, } P(A=a) &= \tbinom{k}{a} p^a (1-p)^{k-a} \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \frac{p^a e^{-\lambda}}{a!} \sum_{k=a}^{\infty} \frac{k!}{(k-a)!} (1-p)^{k-a} \frac{\lambda^k}{k!} \\
&= \frac{p^a e^{-\lambda}}{a!} \sum_{n=0}^{\infty} \frac{(1-p)^n}{n!} \lambda^{n+a} \\
&= \frac{p^a e^{-\lambda}}{a!} \lambda^a \sum_{k=a}^{\infty} \frac{(1-p)^n}{n!} \lambda^n \\
&= \frac{(p\lambda)^a e^{-\lambda}}{a!} e^{\lambda(1-p)} \\
&= \frac{(p\lambda)^a e^{-p\lambda}}{a!}, \text{ which is the pdf of a Poisson r.v. with paramter } p\lambda
\end{aligned}$$

c)

$$\begin{aligned}
p_{A,B}(a,b) &= P(A=a, B=b) \\
&= \sum_{k=0}^{\infty} P(A=a, B=b|N=k)P(N=k) \\
&= (\text{since } P(A=a, B=b|N=k) = 0 \text{ for all } k \neq a+b) \\
&= P(A=a, B=b|N=a+b)P(N=a+b) \\
&= P(A=a|N=a+b)P(N=a+b) \\
&= \tbinom{a+b}{a} p^a (1-p)^b \frac{\lambda^{(a+b)} e^{-\lambda}}{(a+b)!}
\end{aligned}$$

d) Conditioning on N, we have $A = n - B \Rightarrow A, B$ are clearly not independent.

For the unconditional case: We have $p_A(a) = P(A=a) = \frac{(p\lambda)^a}{a!} e^{-p\lambda}$ and $p_B(b) = P(B=b) = \frac{(p\lambda)^b}{b!} e^{-p\lambda}$.

$$\begin{aligned}
\text{Thus, } P(A=a)P(B=b) &= p^a (1-p)^b \lambda^a \lambda^b e^{-\lambda(p+(1-p))} \frac{1}{a!b!} \\
&= p^a (1-p)^b \lambda^{(a+b)} e^{-\lambda} \frac{(a+b)!}{a!(a+b-a)!} \frac{1}{(a+b)!} \\
&= p^a (1-p)^b \lambda^{(a+b)} e^{-\lambda} \tbinom{a+b}{a} \frac{1}{(a+b)!} \\
&= P(A=a, B=b) \Rightarrow A, B \text{ are independent!}
\end{aligned}$$