Name: ____________________________ Student ID No: _______________

UNIVERSITY OF CALIFORNIA

College of Engineering
Department of Electrical Engineering and
Computer Sciences

Professor Tse | Fall 1999

EECS 126 — MIDTERM #2

8 November 1999, 11:10–12:10

Please explain your work carefully. Answers to questions involving Normal rv's can lie left in terms of $\Phi(\cdot)$, the cdf of $N(0, 1)$ rv.

Some pmf's and pdf's:

Bern(p): $P_X(1) = p, P_X(0) = 1 - p \qquad E[X] = p, \mathrm{Var}(X) = p(1-p)$

$M_X(s) = 1 - p + pe^s$

Binomial(n,p): $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n$

$E[X] = np, \mathrm{Var}(X) = np(1-p), M_X(s) = (1 - p + pe^s)^n$

Geometric(p): $P_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots,$

$E[X] = \frac{1}{p}, \mathrm{Var}(X) = \frac{1-p}{p^2}, M_X(s) = \frac{pe^s}{1-(1-p)e^s}$

Poisson(λ): $P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$

$E[X] = \lambda, \mathrm{Var}(X) = \lambda, M_X(s) = e^{\lambda(e^s - 1)}$

Normal(μ, σ^2): $f_X(x) = \frac{1}{\sqrt{2n}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x-\mu)\right]^2$

$E[X] = \mu, \mathrm{Var}(X) = \sigma^2, M_X(s) = \exp\left[\frac{\sigma^2 s^2}{2} + \mu s\right]$

Exponential(λ): $f_X(x) = \lambda e^{-\lambda x}, \qquad x \ge 0$

$E[X] = \frac{1}{\lambda}, \mathrm{Var}(X) = \frac{1}{\lambda^2}, M_X(s) = \frac{\lambda}{\lambda - s} \quad (s > \lambda)$

[25 pts.] 1. X and Y are two independent rv's, uniformly distributed in [0,1]. Let $V = \max(X, Y)$, $W = \min(X, Y)$.

a) Find the pdf's of V and W. (10 pts.)

b) Find $E\left[V | V > \frac{1}{2}\right]$. (5 pts.)

c) Let $U = V - W$. Find the cdf of U. (10 pts.)

[15 pts.] 2. You have available a rv X with pdf $Exponential(\lambda)$. Explain how you can use X to generate:

a) a rv with pdf $\text{Exponential}(\mu)$. (7 pts.)

b) a rv uniformly distributed in [0,1]. (8 pts.)

[30 pts.] 3. The traffic of n users are multiplexed at a switch with outgoing link rate of c bits/s. At time 0, the incoming traffic rate of the i^{th} user is a rv X_i. Data is lost if the **aggregate** incoming traffic rate exceeds the outgoing rate c.

a) Suppose we model the X_i's as iid. $N(\mu, \sigma^2)$. Find the largest number of users we can accommodate such that the probability of data lost is less than 10^{-3}. (10 pts.)

b) Now suppose the behavior of different users is dependent. We model this by having $X_i = Z + Y_i$, where $Z, Y_1, Y_2, \ldots Y_n$ are iid. $N\left(\frac{\mu}{2}, \frac{\sigma^2}{2}\right)$ rv's. Find the mean and variance of X_i. Find the covariance between X_i and X_j. (10 pts.)

c) Find the maximum number of users that can be accommodated in a link of rate c, such that the probability of data loss is less than 10^{-3}. How does this number compare to the answer to (a)? Give some intuition to support your answer. (10 pts.)

[30 pts.] 4. Packets arriving at a switch are routed to either destination A (with probability p) or destination B (with probability $1 - p$). The destination of each packet is chosen independently of each other. In time interval [0,1], the number of arriving packets is $Poisson(\lambda)$.

a) Find the expected number of packets routed to A in time interval [0,1]. (5 pts.)

b) Show that the number of packets routed to A is Poisson distributed. With what parameter? (Hint: Express the number as a sum of random number of rv's.) (10 pts.)

c) Find the joint pmf of the number of packets routed to A and the number of packets routed to B. (Hint: You may want to first condition on an appropriate event.) (10 pts.)

d) Are the number of packets routed to A and to B independent, conditional on the total number of arriving packets being n? Are the number of packets routed to A and to B independent? (5 pts.)