Problem #1
Many network protocols require an estimate of the mean roundtrip time of a connection between the sender and receiver. (The roundtrip time is the time for a packet to get to the receiver and return to the sender.) The mean roundtrip time depends on things like the number of user currently on the network and the route of the connection.

One way of obtaining an estimate is by sending probe packets. Suppose X1, X2, ...Xn are the roundtrip times experienced by these packets; they are modeled as noisy measurements of the (unknown) mean roundtrip time \( \mu \):

\[
X_i = \mu + Z_i, \quad i = 1, ..., n
\]

where \( Z_i \) are i.i.d. \( N(0, \omega^2) \) random variables. Consider now two estimators of \( \mu \):

\[
U_n = \frac{1}{n}(\text{Summation} (1\rightarrow n): X_i)
\]

\[
V_n = c(\text{Summation} (1\rightarrow n): \alpha^{n-i}X_i)
\]

where \( \alpha \) is a positive constant less than 1 and \( c \) is some constant to be chosen. \( V_n \) is an exponential weighting of the past measurements.

a. An estimator \( \mu(n) \) of \( \mu \) is said to be unbiased if \( E(\mu(n)) = \mu \). Is \( U_n \) unbiased? Choose the constant \( c \) such that \( V_n \) is unbiased. (6 pts.)

b. A sequence of unbiased estimators \{\( \mu(n) \)\} is said to be consistent if \( \lim n\rightarrow\infty \text{var}(\mu(n)) = 0 \). Are \{\( U_n \)\} and \{\( V_n \)\} consistent? Give an intuitive explanation. (8 pts.)

c. In practice, the estimator \( V_n \) is often preferred over \( U_n \). Can you give a reason that is not captured by our model? (4 pts.)

Problem #2
We wish to multiplex voice calls onto a switch with outgoing link speed of \( C \) packets per time slot. At any time, each voice call has a probability \( p \) of being active and \( 1-p \) of being silent. When it's active, it sends packets at a rate of 1 packet/time slot; when it's silent, it sends no packets. We wish to determine the number of voice calls that can be accommodated by the switch.

a. Suppose we can tolerate no packet loss from the system. What is the maximum number of calls we can accept in the system? (2pts)
b. Suppose now that we can tolerate a probability alpha of the event that more than a total of C packets arrive at the switch at a time slot. (alpha typically small, say $10^{-3}$). If the link speed C is high, give a good approximation of the maximum number of calls acceptable in the system. Is this number a random variable? (10 pts)

c. Call the ratio of the number in (b) to the number in (a) the statistical multiplexing gain G. This gain depends on parameters C, p, and alpha. Consider three scenarios:
   i) alpha -> 1
   ii) alpha -> 0
   iii) C -> infinity

What does G approach in each of the 3 cases? Intuitive justification is sufficient; detailed calculations not required. For (ii), reasoning based on the approximation in (b) may not give you the right answer. Why? (6 pts)

**Problem #3**

The sequence $X_1, X_2, \ldots$ represents speech samples that we want to quantize using a 1-bit A/D (analog-to-digital) converter. We model each sample as an $N(0, \omega^2)$ random variable.

a. Consider first a strategy where we quantize each sample individually. The quantizer has the form:

$q(x) = a$ if $x > 0$
$q(x) = -a$ if $x$

Find the value of $a$ to minimize the mean-square error $E[(q(X_i) - X_i)^2]$ and the corresponding minimum value. (Hint: Condition on an appropriate event.) Does it matter what value $q(.)$ maps $x=0$ to? (8 pts)

b. Now suppose consecutive samples are correlated with correlation coefficient rho. Instead of quantizing the $X_i$'s, we quantize $Y_i = X_i - X'_i$ where $X'_i$ is the MMSE estimate of $X_i$ based on $X(i-1)$. Using the same form of quantizer as in part (a), find the value of $a$ that minimizes $E[(q(Y_i) - Y_i)^2]$. For what values of rho would you expect to see an improvement over the strategy in (a)? (8 pts)

**Problem #4**

Alice wants to deliver a packet to Bob over a network. The time to get there is random, exponentially distributed with mean $\mu$. If Bob gets the packet within a time $d$, he is happy and will give Alice $y$ dollars. If not, Bob will pay nothing.

a. Just to increase the chance that Bob will get the packet within time $d$, Alice decides to make $n$ copies of the packet and send them simultaneously. If the times for each packet to get to the destination is i.i.d. with the above distribution, find the probability that at least one copy will get to Bob. (6 pts)

b. Alice is charged $1 for each copy of the packet sent as a penalty for congesting teh network. Find the optimal number of copies Alice should make to maximize her expected profit. (Note that Bob will pay only $y$ dollars no matter how many copies he receives.) (4 pts.)

c. Suppose now that Alice has available feedback information from Bob, so that at any time she knows whether Bob has received the packet or not. So she decides to try the following strategy instead. First at time 0, she sends one copy of the packet. If by time $\tau < d$, Bob has not received this packet copy, she will send
another copy. Calculate the expected profit under this strategy. (6 pts)

d. What do you think is the optimal strategy to maximize the expected profit if d is very large? (4 pts)

**Problem #5**

Suppose $X_1, X_2, \ldots, X_n$ are independent $N(\mu, \sigma^2)$ random variables.

a. Suppose we know $\sigma^2$ and we want to estimate $\mu$ from the observations $X_1, \ldots, X_n$.

i) Find the maximum likelihood (ML) estimate $\mu'$. (4 pts)

ii) We are interested in selecting a beta such that the event $\mu \in [\mu' - \beta \sigma, \mu' + \beta \sigma]$ happens with probability $1 - \alpha$ (\alpha close to 0). Explain clearly why this is an event of a random experiment. Compute $\beta$. (4 pts)

b. Suppose now both $\mu$ and $\sigma$ are unknown. It turns out that in this case, the ML estimate of $\mu$ remains the same as in part (a), and the ML estimate of $\sigma$ is given by

$$\sigma' = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \mu')^2}$$

i) A natural $(1 - \alpha) \times 100\%$ confidence interval for the parameter $\mu$ when $\sigma$ is unknown is $I = [\mu' - \beta \sigma', \mu' + \beta \sigma']$

with $\beta$ chosen such that

$$\Pr(\mu \in I) = 1 - \alpha$$

Let $Y \Delta = (\mu' - \mu)/\sigma'$. Compute $\beta$ in terms of the cdf of the random variable $Y$. Does the cdf of $Y$ depend on the true values of the parameters $\mu$ and $\sigma^2$? Why is this important? (6 pts)

ii) Argue that $\mu'$ and $X_i - \mu'$ are jointly Gaussian. (6 pts)

iii) Show that $\mu'$ and $X_i - \mu'$ are uncorrelated. (4 pts)

iv) Using (i) and (ii) or otherwise, show that $\mu'$ and $\sigma'$ are independent. (4 pts)

v) (Bonus) Consider the special case $n = 2$. Using part (iv) or otherwise, show that the pdf of $Y \Delta = (\mu' - \mu)/\sigma'$ is

$$f_Y(y) = \frac{\sqrt{2}}{\pi(1 + 2y^2)}.$$ 

This allows us to compute explicitly the confidence interval in (b) (ii). (8 pts)
If you have any questions about these online exams please contact examfile@hkn.eecs.berkeley.edu.