UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Process

Solution 10
Fall 2014

Issued: Monday, November 20, 2014

Self-graded Scores Due: 5:00pm Monday, November 24, 2014
Submit your self-graded scores via the google form: http://goo.gl/rFeFBa.
Make sure that you use your **SORTABLE NAME** on bCourses.

Problem 1.

Midterm #2 - Problem 1. (a) We have

$$\begin{aligned}\Pr(X > 4) &= \Pr(X - 1 > 3) \\ &= \frac{1}{2}\Pr(|X - 1| > 3) \\ &= \frac{1}{2} \times 2/9 = 1/9.\end{aligned}$$

(b) They have same distributions. The reason is that $\Pr(Y_1 \leq y) = \Pr(-\sqrt{y} \leq X_1 \leq \sqrt{y}) = \sqrt{y}$. Also $\Pr(Y_2 \leq y) = \Pr(X_2 \leq \sqrt{y}) = \sqrt{y}$.

(c) With probability 1/3 you arrive at a time that its corresponding interarrival time is 15 min and you have an expected waiting time of 7.5 min, and with probability 2/3 you arrive at a time that its corresponding interarrival time is 30 min and you have expected waiting time of 15 min. So, the expected waiting time is $2/3 \times 15 + 1/3 \times 7.5 = 37.5/3 = 12.5$.

(d) We need to find expected value and variance of T_100. We have

$$E[T_{100}] = 100 \quad \text{and} \quad var(T_{100}) = 100.$$

So we approximate the distribution with $Z \sim N(100, 100)$. Then,

$$\Pr(Z > 110) = \Pr(\frac{Z - 100}{10} > 1) = Q(1) \simeq 0.16.$$

(e) Let $\lambda = \lambda_n + \lambda_s + \lambda_w + \lambda_e$. Then, an arriving car is coming from north with probability $p_n = \lambda_n/\lambda$, from south with probability $p_s = \lambda_s/\lambda$ and so on. Now given 8 arrivals, the probability of 2 cars from each direction is

$$\frac{8!}{2!.2!.2!.2!} p_n^2 p_s^2 p_w^2 p_e^2.$$

Midterm #2 - Problem 2. (a) It is easy to see that the answer is $0.6 \times 0.4 + 0.4 \times 0.5 = 0.44$.

(b) We solve flow-conserving equations

$$\pi(1)0.4 = \pi(2)0.2$$
$$\pi(2)0.3 = \pi(3)0.1$$
$$\pi(1) + \pi(2) + \pi(3) = 1.$$

So $\pi = [1/9, 2/9, 6/9]$.

(c) The limit is $\pi(1) \times 0.4 + \pi(2) \times 0.3 = 1/9$.

(d) No. Note that $\Pr(Y_n = 1|Y_{n-1} = 1, Y_{n-2} = 1) = 0$ but $\Pr(Y_n = 1|Y_{n-1} = 1) > 0$.

(e) In the stationary case, the probability of a right transition is $\pi(1)0.4+\pi(2)0.3$. So the answer is

$$\frac{\pi(1)0.4}{\pi(1)0.4 + \pi(2)0.3}.$$

(f) We write first step equations. Let β be the expected hitting time of 1 from 2 and γ be the expected hitting time of 3. Then,

$$\beta = 1 + 0.5\beta + 0.3\gamma$$
$$\gamma = 1 + 0.9\gamma + 0.1\beta.$$

Then, $E(T)$ is the expected time to hit state 2 plus β, which is $\beta + 1/0.4$.

Midterm #2 - Problem 3. (a) Let t be the number of right moves in our samples. Then, probability of the sample sequence is $p^t(1-p)^{n-t}$. Taking the log and setting derivative equal to 0 we have

$$t/p - (n-t)/(1-p) = 0 \Rightarrow p_{ML} = t/n.$$

The estimator is unbiased since $E[T/n] = p$, where T is the random variable denoting the number of right moves.

(b) The number of right moves in n moves is binomial with parameters n and p which can be approximated by $Z \sim N(np, np(1-p))$. Since the likelihood function is monotone we consider the test $\hat{X} = 1\{Z > \tau\}$. We want to find τ such that

$$\Pr(Z > \tau) \leq 0.002 \; \forall p,$$

so we consider the worst-case $p = 0.5$. Then, we have

$$Q(\frac{\tau - 0.5n}{\sqrt{n/4}}) = 0.002 = Q(2.88).$$

Thus, $\tau = 1.44\sqrt{n} + 0.5n = 20288$.

(c) Let t_1 be the number of right moves and t_2 be the number of left moves. Then, the probability of a sample random walk is $p^{t_1}q^{t_2}(1-p-q)^{n-t_1-t_2}$. Again we take log of the function and set the partial derivative with respect to p and q to 0. Then,

$$\frac{t_1}{p} - \frac{n-t_1-t_2}{1-p-q} = 0$$
$$\frac{t_2}{q} - \frac{n-t_1-t_2}{1-p-q} = 0.$$

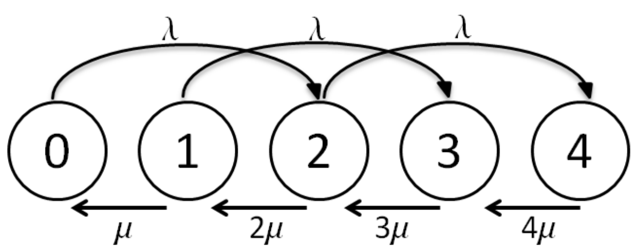
So $p_{ML} = t_1/n$ and $q_{ML} = t_2/n$.

Midterm #2 - Problem 4. (a) Due to phase transition effect, as n goes to infinity, given that $Y = 1$, any value $p \in (\log(n)/n, 1]$ is a maximum-likelihood estimator. Given that $Y = 0$, any value $p \in [0, \log(n)/n)$ is the MLE.

(b) Given that $Y = 1$, we know that $p > \log(n)/n$ so H_1 will be detected. Given that $Y = 0$, all values $p \in [0, \log(n)/n)$ are equally likely so we can randomize the decision as follows. With probability γ we detect H_1. Then,

$$\Pr(H_1 \text{ detected}|H_0) = \gamma = 0.1.$$

(c) Given the prior the MAP estimator of p given $Y = 1$ is $2/3$ and the MAP estimator of p given $Y = 0$ is 0.



Midterm #2 - Problem 5. (a) The arrival transitions from state i to state $i+2$ is λ for $0 \le i \le 2$. The service transitions from state i to state $i-1$ is $i\mu$ for $1 \le i \le 4$.

(b) The blocking probability is $\pi_3 + \pi_4$ because a job cannot find 2 (or more) servers with this probability.

(c) The average download time for two chunks is $\frac{1}{2\mu} + \frac{1}{\mu} = 1.5\frac{1}{\mu}$.

(d) The coded storage is more flexible because it can accept a job if the number of busy disks is less than or equal to 2 regardless of which disks are busy. On the other hand, the uncoded storage is less flexible. For instance, it cannot accept a job if both disks that are storing F_1 are busy because the job cannot find a disk that can provide F_1.

(e) Using the conservation of flow,

$$\begin{aligned}
\lambda\pi_0 &= \pi_1 \\
\lambda(\pi_0 + \pi_1) &= 2\pi_2 \\
\lambda(\pi_1 + \pi_2) &= 3\pi_3 \\
\lambda\pi_2 &= 4\pi_4
\end{aligned}$$

From the first and the second equations, $\pi_2 = \frac{\lambda(\lambda+1)}{2}\pi_0$. From the third equation, $\pi_3 = \frac{\lambda^2(\lambda+3)}{6}\pi_0$. From the last equation, $\pi_4 = \frac{\lambda^2(\lambda+1)}{8}\pi_0$. By normalizing these using $\sum \pi_i = 1$, we obatin

$$\pi = \frac{24}{7\lambda^3 + 27\lambda^2 + 36\lambda + 24}[1, \lambda, \frac{\lambda(\lambda+1)}{2}, \frac{\lambda^2(\lambda+3)}{6}, \frac{\lambda^2(\lambda+1)}{8}].$$