SOLUTIONS

Formulas: Given the short attention span induced by twitter and the like, we thought you might appreciate not having to remember the following formulas. After all, they are on Wikipedia.

\[
X = N(\mu, \Sigma) \iff f_X(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(x - \mu)^T\Sigma^{-1}(x - \mu)\}
\]

\[(X, Y) \text{ J.G. } \Rightarrow E[X|Y] = E(X) + \Sigma_{XY}\Sigma_Y^{-1}(Y - E(Y))
\]

\[\text{cov}(AX, BY) = A \text{cov}(X, Y) B^T.\]

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]

If \( V = N(0, 1) \), then \( P(V > 1) = 0.159, P(V > 1.64) = 0.05, \)

\[
P(V > 1.96) = 0.025, P(V > 2) = 0.023, P(V > 2.58) = 0.005.
\]

Problem 1. (Short Problems 40%)

- Give an example where \( E[X|Y] = E(X) \) but \( X, Y \) are not independent.
  
  Let \( X = YZ \) where \( Y, Z \) are independent and uniform in \([-1, 1] \).

- Let \( X, Y \) be i.i.d., \( B(100, 0.3) \). Calculate \( E[X - Y|X + Y] \).
  
  By symmetry, \( E[X - Y|X + Y] = 0.\)

- Let \( X, Y \) be as in the previous problem. Calculate \( E[(X + Y)^2|X] \).

  We have

  \[
  E[(X + Y)^2|X] = X^2 + 2XE(Y) + E(Y^2) = X^2 + 60X + \text{var}(Y) + E(Y)^2
  \]

  \[
  = X^2 + 60X + 100 \times 0.3 \times 0.7 + (30)^2
  \]
• Assume that \( \Sigma_X = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \). Calculate \( \text{var}(2X_1 + 3X_2 + X_3) \).

One has \( \text{var}(a^T X) = a^T \Sigma_X a \). So,

\[
\text{var}(2, 3, 1X) = [2, 3, 1] \begin{bmatrix} 4 & -1 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = [3, 4, -3] \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 13.
\]

• Let \( X, Y, Z \) be i.i.d. \( U[0, 1] \). Calculate \( E[2X + 3Y + 4Z|X + Y + Z] \).

By symmetry,

\[
\]

Hence,

\[
E[2X + 3Y + 4Z|X + Y + Z] = (2 + 3 + 4) \times \frac{1}{3}(X + Y + Z) = 3(X + Y + Z).
\]
Problem 2. (20%) Assume that \( Y = X + aZ \) where \( X, Z \) are i.i.d., \( N(0, 1) \).

(a) Calculate \( E[X|Y] \);

(b) What is the variance of \( X \) given \( Y \)?

(c) Given \( Y = y \), what is the distribution of \( X \)?

(d) Express \( P[X > c|Y = y] \) in terms of \( \Phi(z) := P(Z \leq z) \) where \( Z = N(0, 1) \).

(a) \( E[X|Y] = \Sigma_{X,Y} \Sigma_{Y}^{-1} Y = Y/(1 + a^2) =: bY \).

(b) We know that \( X - E[X|Y] = X - bY = X - b(X + aZ) = (1 - b)X - abZ \), so that the variance of \( X \) given \( Y \) is equal to

\[
\sigma^2 = \text{var}((1 - b)X - abZ) = (1 - b)^2 + (ab)^2 = \frac{a^2}{1 + a^2}.
\]

(c) Given \( Y = y \), \( X = N(by, \sigma^2) \).

(d) We have

\[
P[X > c|Y = y] = P(N(by, \sigma^2) > c) = P(N(0, \sigma^2) > c - by)
= P(N(0, 1) > \frac{c - by}{\sigma}) = 1 - \Phi(\frac{c - by}{\sigma}).
\]
Problem 3. (20%) Assume that $Y = 4X + Z$ where $Z = N(0, 1)$ and $P(X = k) = 1/3$ for $k \in \{-1, 0, 1\}$.

a) Calculate $\hat{X} = MAP[X|Y]$.

b) Calculate $P(\hat{X} \neq X)$.

a) A sketch of the densities shows that

$$MAP[X|Y] = \begin{cases} 
-1, & \text{if } Y \leq -2 \\
0, & \text{if } Y \in (-2, 2) \\
1, & \text{if } Y \geq 2.
\end{cases}$$

b) We find,

$$P(\hat{X} \neq X) = \frac{1}{3}P[\hat{X} \neq X]|X = -1] + \frac{1}{3}P[\hat{X} \neq X]|X = 0] + \frac{1}{3}P[\hat{X} \neq X]|X = 1]$$

$$= \frac{1}{3}P[Y \geq -1|X = -1] + \frac{1}{3}P[Y < -1 \text{ or } Y > 1|X = 0] + \frac{1}{3}P[Y < 1|X = 1]$$

$$= \frac{1}{3}P(Z \geq 2) + \frac{1}{3}P(Z < -2 \text{ or } Z > 2) + \frac{1}{3}P(Z < -2)$$

$$= \frac{1}{3}[\alpha + 2\alpha + \alpha] = \frac{4}{3}\alpha$$

where $\alpha = P(Z > 2) = 0.023$. Thus,

$$P(\hat{X} \neq X) \approx 3\%.$$
Problem 4. (20%) When $X = 0$, $Y = N(0, 1)$. When $X = 1$, $Y = N(0, 4)$.

(a) Find $\hat{X} = g(Y)$ that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 5%$.

(b) What is $P[\hat{X} = 1|X = 1]$?

(a) One has

$$\hat{X} = 1 \text{ iff } L(y) \geq \lambda$$

where $\lambda$ is such that $P[\hat{X} = 1|X = 0] = 5\%$. Now,

$$L(y) = \frac{f_1(y)}{f_0(y)} \text{ with } f_1(y) = \frac{1}{\sqrt{2\pi}4} \exp\{-y^2/8\}, f_0(y) = \frac{1}{\sqrt{2\pi}} \exp\{-y^2/2\}.$$ 

Hence,

$$L(y) = A \exp\{y^2(1/2 - 1/8)\} = A \exp\{3y^2/8\}.$$ 

Thus,

$$L(y) \geq \lambda \iff |y| > \alpha,$n

for some $\alpha$.

We should choose $\alpha$ so that

$$P[\hat{X} = 1|X = 0] = P(|Y| > \alpha|X = 0] = P(|N(0, 1)| > \alpha) = 5\%.$$ 

Hence, $\alpha = 1.96$.

(b) One finds

$$P[\hat{X} = 1|X = 1] = P(|Y| > \alpha|X = 1 = P(|N(0, 4)| > 1.96) \approx P(|N(0, 1)| > 1) \approx 32\%.$$