Department of EECS - University of California at Berkeley

EECS 126 - Probability and Random Processes - Fall 2008

Midterm 2: 11/18/2008

SOLUTIONS

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1. Definition (10%)

Define "Jointly Gaussian Random Variables"

Answer. *A collection of random variables with the property that an arbitrary linear combination of them is Gaussian.*

Also acceptable: A collection of random variables that are linear combinations of i.i.d. standard Gaussian random variables.

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2. Orthogonality (10%)

Give an example of a two orthogonal random variables that are not independent.

Answer. *Let* (X, Y) *be uniformly distributed in the set* $\{(-1,0),(0,-1),(1,0),(0,1)\}$.

3. Gaussian but not jointly (10%)

Give an example of two $N(0,1)$ random variables that are not jointly Gaussian.

Answer. *Let* X, Z *be independent with* $X = N(0,1)$ *and* $P(Z = -1) = P(Z = 1)$. *Then* X *and* $Y = XZ$ *are* $N(0,1)$ *but not jointly Gaussian.*

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4. Conditional Expectation (10%)

Is it true that $E[X|Y] = 0$ implies that X and Y are uncorrelated? Prove or provide a counterexample.

Answer. *This is true. First note that* $E(X) = E(E[X|Y]) = 0$. *Hence,*

$$E(XY) = E(E[XY|Y]) = E(YE[X|Y]) = E(Y.0) = 0 = E(X)E(Y).$$

5. Conditional Expectation, again (10%)

Let X, Y, Z be i.i.d. and uniformly distributed in $[0,1]$. Calculate $E[(X+Y)^2|Y+Z]$.

Answer. *First note that*

$$E[(X+Y)^2|Y+Z] = E[X^2+2XY+Y^2|Y+Z] = \frac{1}{3}+E[Y+Y^2|Y+Z] = \frac{1}{3}+\frac{1}{2}(Y+Z)+E[Y^2|Y+Z].$$

By drawing Figure 1, we see that, given $Y+Z=u$, Y is uniform in $[0,u]$ if $0<u<1$ and Y is uniform in $[u-1,1]$ if $1<u<2$. Also, note that if $Y=U[a,b]$, then

$$E(Y^2) = \int_a^b y^2 \frac{1}{b-a} dy = \frac{b^3-a^3}{b-a} = \frac{1}{3}(a^2+ab+b^2).$$

Hence,

$$E[Y^2|Y+Z=u] = \begin{cases} u^2/3, & \text{if } 0<u<1, \\ (u^2-u+1)/3, & \text{if } 1<u<2. \end{cases}$$

Finally, putting the pieces together,

$$E[Y^2|Y+Z=u] = \begin{cases} 1/3+(Y+Z)/2+(Y+Z)^2/3, & \text{if } 0<Y+Z<1, \\ 1/3+(Y+Z)/2+((Y+Z)^2-(Y+Z)+1)/3, & \text{if } 1<Y+Z<2. \end{cases}$$

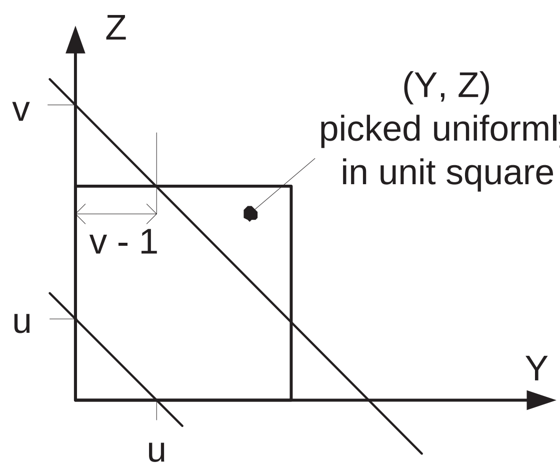

Figure 1: Finding the density of Y given $Y+Z$.

6. Flipping coins (10%)

You flip a coin n times. The probability p that a coin toss yields H is uniformly distributed in $[0, 1]$. Calculate the variance of the number of Hs in the n tosses.

Answer. *Let X be the number of Hs. Then*

$$E(X) = E(E[X|p]) = E(np) = n/2.$$

Also,

$$E(X^2) = E(E[X^2|p]) = E(var[X|p]+E[X|p]^2) = E(np(1-p)+(np)^2) = E(np-np^2+n^2p^2) = \frac{n}{2}-\frac{n}{3}+\frac{n^2}{3}.$$

Hence,

$$var(X) = E(X^2) - E(X)^2 = [\frac{n}{2} - \frac{n}{3} + \frac{n^2}{3}] - \frac{n^2}{4} = \frac{n(2+n)}{12}.$$

7. Jointly Gaussian (15%)

Let (X, Y) be jointly Gaussian, zero mean, with $\text{var}(X) = 4$, $\text{var}(Y) = 1$ and $\text{cov}(X, Y) = 1$. Calculate $E[X^2|Y]$.

Answer. *Recall that*

$$E[X|Y] = \frac{cov(X, Y)}{var(Y)} Y = Y,$$

so that $X = Y + Z$ *where* $Z = X - Y$ *is independent of* Y. *Also,* $var(Z) = E((X - Y)^2) = 4 - 2 + 1 = 3$.

Hence, given Y, $X = N(Y, 3)$. *Now, if* $V = N(\mu, \sigma^2)$, *we see that* $E(V^2) = \mu^2 + \sigma^2$. *It follows that*

$$E[X^2|Y] = Y^2 + 3.$$

8. Jointly Gaussian, again (15%)

Assume that $(X, Y_1, Y_2)^T = N(\mathbf{m}, \Sigma)$ with

$$\mathbf{m} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Calculate $E[X|Y_1, Y_2]$.

Answer. *Let* $\mathbf{Y} = [Y_1, Y_2]^T$. *Then*

$$\begin{aligned} E[X|\mathbf{Y}] &= E(X) + \Sigma_{X,\mathbf{Y}} \Sigma_{\mathbf{Y}}^{-1} \{\mathbf{Y} - E(\mathbf{Y})\} \\ &= 3 + [1, 2] \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \{\mathbf{Y} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}\} \\ &= 3 + [1, 2] \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \{\mathbf{Y} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}\} \\ &= 3 + [-1, 3]\{\mathbf{Y} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}\} = 8 - Y_1 + 3Y_2. \end{aligned}$$

9. Detection and Hypothesis Testing (10%)

Given $X \in \{0, 1\}$, the random variable Y is exponentially distributed with rate $3X + 1$ (thus, with mean $(3X + 1)^{-1}$).

1) Assume $P(X = 1) = p, P(X = 0) = 1 - p$. Find the MAP estimate of X given Y.

2) Find the MLE of X given Y.

3) Solve the hypothesis testing problem of X given Y with a probability of false alarm at most 10%. That is, find $\hat{X}$ as a function of Y that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 0.1$.

4) For what value of p does one have the same solution for 1) and 3)?

Answer. *We start by calculating the likelihood ratio $L(Y)$. Let $f_1(y)$ be the density of Y when $X = 1$ and $f_0(y)$ be its density when $X = 0$. We find*

$$L(y) = \frac{f_1(y)}{f_0(y)} = \frac{4\exp\{-4y\}}{1\exp\{-y\}} = 4\exp\{-3y\}.$$

We note that $L(y)$ is decreasing in y, so that $L(Y) > \lambda \Leftrightarrow Y < y_0$.

1) $MAP[X|Y] = 1\{Y < y_0\}$ where y_0 is such that $L(y_0) = (1-p)/p$. That is,

$$4\exp\{-3y_0\} = \frac{1-p}{p},$$

which yields $y_0 = \frac{1}{3}\ln(\frac{4p}{1-p})$.

2) $MLE[X|Y] = 1\{Y < y_0\}$ where y_0 is as before, but with $p = 1/2$. That is $y_0 = \frac{1}{3}\ln(4)$.

3) $HT[X|Y] = 1\{Y < y_0\}$ where y_0 is such that

$$0.1 = P[Y \leq y_0|X = 0] = 1 - \exp\{-y_0\}.$$

Hence, $y_0 = -\ln(0.9)$.

4) The solutions of 1) and 3) coincide if

$$\frac{1}{3}\ln(\frac{4p}{1-p}) = -\ln(0.9)$$

which is seen to happen if $p = 1/(1 + 4(0.9)^3) \approx 0.255$.