1. **Definition (10%)**
Define “Jointly Gaussian Random Variables”

2. **Orthogonality (10%)**
Give an example of a two orthogonal random variables that are not independent.
3. Gaussian but not jointly (10%)
Give an example of two $N(0,1)$ random variables that are not jointly Gaussian.

4. Conditional Expectation (10%)
Is it true that $E[X|Y] = 0$ implies that $X$ and $Y$ are uncorrelated? Prove or provide a counterexample.
5. Conditional Expectation, again (10%)

Let $X, Y, Z$ be i.i.d. and uniformly distributed in $[0, 1]$. Calculate $E[(X + Y)^2|Y + Z]$. 
6. Flipping coins (10%)

You flip a coin \( n \) times. The probability \( p \) that a coin toss yields \( H \) is uniformly distributed in \([0, 1]\). Calculate the variance of the number of \( H \)s in the \( n \) tosses.
7. Jointly Gaussian (15%)

Let \((X, Y)\) be jointly Gaussian, zero mean, with \(\text{var}(X) = 4\), \(\text{var}(Y) = 1\) and \(\text{cov}(X, Y) = 1\). Calculate \(E[X^2|Y]\).
8. Jointly Gaussian, again (15%)

Assume that \((X, Y_1, Y_2)^T = N(m, \Sigma)\) with

\[
m = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 6 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.
\]

Calculate \(E[X|Y_1, Y_2]\).
9. Detection and Hypothesis Testing (10%)  
Given $X \in \{0, 1\}$, the random variable $Y$ is exponentially distributed with rate $3X + 1$ (thus, with mean $(3X + 1)^{-1}$).

1) Assume $P(X = 1) = p, P(X = 0) = 1 - p$. Find the MAP estimate of $X$ given $Y$.

2) Find the MLE of $X$ given $Y$.

3) Solve the hypothesis testing problem of $X$ given $Y$ with a probability of false alarm at most 10%. That is, find $\hat{X}$ as a function of $Y$ that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 0.1$.

4) For what value of $p$ does one have the same solution for 1) and 3)?