1. Short Questions (20%); 4% each

1.1. Define “Random Variable”

*Answer:*

A random variable is a real-valued function of the outcome or a random experiment (80%). Details: To specify the random experiment, one defines a probability space (set of outcomes $\Omega$ and probability $P(A)$ of sets $A$ of outcomes). The random variable is then a function $X : \Omega \rightarrow \mathbb{R}$ (20%).

1.2. Complete the sentence: $A, B, C$ are mutually independent if and only if

*Answer:*

\[
P(A \cap B) = P(A)P(B), \quad P(B \cap C) = P(B)P(C), \quad P(A \cap C) = P(A)P(C),
\]

\[
P(A \cap B \cap C) = P(A)P(B)P(C).
\]
1.3. Bayes’s Rule. Assume that \( \{A_1, \ldots, A_n\} \) form a partition of \( \Omega \) and that \( p_m = P(A_m) \) and \( q_m = P[B|A_m] \) for \( m = 1, \ldots, n \). Derive \( P[A_m|B] \) in terms of \( p \)'s and \( q \)'s.

**Answer:**

\[
P[A_m|B] = \frac{P(A_m \cap B)}{P(B)} = \frac{P(A_m)P[B|A_m]}{\sum_{k=1}^{n} P(A_k \cap B)} = \frac{P(A_m)P[B|A_m]}{\sum_{k=1}^{n} P(A_k)P[B|A_k]} = \frac{p_m q_m}{\sum_{k=1}^{n} p_k q_k}.
\]

1.4. Assume that \( X \) is equal to 2 with probability 0.4 and is uniformly distributed in \([0, 1]\) otherwise. Calculate \( E(X) \) and \( \text{var}(X) \). (Hint: Recall that \( \text{var}(X) = E(X^2) - E(X)^2 \).)

**Answer:**

\[
E(X) = 0.4 \times 2 + 0.6 \times 0.5 = 1.1.
\]

Also,

\[
E(X^2) = 0.4 \times 2^2 + 0.6 \times \int_0^1 x^2 \, dx = 0.4 \times 4 + 0.6 \times \frac{1}{3} = 1.8.
\]

Hence,

\[
\text{var}(X) = 1.8 - (1.1)^2 = 0.59.
\]

1.5. Two random variables \( X, Y \) are related so that \( aX + Y = b \) for some real constants \( a \) and \( b \). Given \( E(X) = \mu, \text{var}(X) = \sigma^2 \), express \( E(Y) \) and \( \text{var}(Y) \) in terms of \( \mu \) and \( \sigma \).

**Answer:**

\[
Y = b - aX. \text{ Therefore:}
\]

\[
\]

\[
\text{var}(Y) = E[(Y - E[Y])^2] = E[(b - aX - b + aE[X])^2] = E[a^2(X - E[X])^2] = a^2 \text{var}(X) = a^2 \sigma^2.
\]
2. Posterior Probability (10%)

Two manufacturing plants produce similar parts. Plant 1 produces 1000 parts, 100 of which are defective. Plant 2 produce 2000 parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from plant 1?

Answer:

Define the following events:

- \( A \) = the part is selected from Plant 1
- \( D \) = the part is defective

We wish to find \( P(A \mid D) \). The problem statement gives us \( P(A) = \frac{1}{10} \), \( P(D \mid A) = \frac{1}{10} \), and \( P(D \mid A^c) = \frac{3}{40} \). Bayes’ rule:

\[
P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D \mid A)P(A)}{P(D)}
\]

Since \( \{A, A^c\} \) partition \( \Omega \), we can use the total probability theorem to find \( P(D) \):

\[
P(D) = P(D \mid A)P(A) + P(D \mid A^c)P(A^c)
\]

\[
= \frac{1}{10} \cdot \frac{1}{3} + \frac{3}{40} \left( 1 - \frac{1}{3} \right) = \frac{1}{12}
\]

\[
P(A \mid D) = \frac{\frac{1}{10} \cdot \frac{1}{3}}{\frac{1}{12}} = \frac{2}{5}
\]

3. Posterior Probability (10%)

A number is selected at random from 1,2,...,100. Given that the number selected is divisible by 2, what is the probability that the number is divisible by 3 or 5?

Answer:

Define the following events:

- \( B \) = the number is divisible by 2
- \( C \) = the number is divisible by 3
- \( E \) = the number is divisible by 5

\[
P(C \cup E \mid B) = \frac{P((C \cup E) \cap B)}{P(B)}
\]

\[
= \frac{P(C \cap B) + P(E \cap B) - P(C \cap E \cap B)}{P(B)}
\]

\[
= \frac{\frac{16}{100} + \frac{10}{100} - \frac{3}{100}}{\frac{50}{100}}
\]

\[
= \frac{23}{50}
\]
4. Expectation (15%)

There is a series of mutually independent Bernoulli experiments that individually have probability 
$p$ of success and probability $(1 - p)$ of failure. These experiments are conducted until the $r^{th}$
success. Let $X$ be the number of failures that occur until this $r^{th}$ success. The pmf of $X$ is:

$$p_X(k) = \binom{k + r - 1}{k} p^r (1 - p)^k, k \geq 0$$

a) Justify the pmf.

**Answer:**

$X = k$ if $k$ failures and $r - 1$ successes occur in any order, and then the next experiment is
a success. There are $\binom{k + r - 1}{k}$ possible orderings where this occurs, and each permutation has
probability $p^r (1 - p)^k$. Therefore:

$$p_X(k) = P(X = k) = \binom{k + r - 1}{k} p^r (1 - p)^k$$

b) Express $E(X)$ in terms of $p$ and $r$.

**Answer:**

Using the definition of expectation:

$$E[X] = \sum_{k=0}^{\infty} kp_X(k)$$

$$= \sum_{k=0}^{\infty} k \binom{k + r - 1}{k} p^r (1 - p)^k$$

Noticing that $k$ will cancel part of $k!$, we relate this to $p_X(k - 1; r + 1)$, that is the probability
that $X = k - 1$ in a similar experiment to $r + 1$ successes.

$$E[X] = \sum_{k=0}^{\infty} \binom{k + r - 1}{k} p^r (1 - p)^{k-1} \cdot r \frac{1 - p}{p}$$

$$= r \frac{1 - p}{p} \sum_{k=0}^{\infty} p_X(k - 1; r + 1)$$

$$= r \frac{1 - p}{p}$$
5. Independence (15%)

Show that if three events $A, B$, and $C$ are mutually independent, then $A$ and $B \cup C$ are independent.

\textit{Answer:}
Events $A$ and $B \cup C$ are independent if and only if $P(A \cap (B \cup C)) = P(A)P(B \cup C)$.

\[P(B \cup C) = P(B) + P(C) - P(B \cap C)\]
\[= P(B) + P(C) - P(B)P(C)\]

because $B$ and $C$ are independent.

\[P(A)P(B \cup C) = P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)\]
\[= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)\]
\[= P((A \cap B) \cup (A \cap C))\]
\[= P(A \cap (B \cup C))\]

6. Probability Distribution (10%)

Find the allowable range of values for constants $a$ and $b$ such that the following function is a valid CDF

\[F(x) = 1 - ae^{-x/b} \text{ if } x \geq 0, \text{ and } 0 \text{ otherwise.}\]

For those values of $a$ and $b$, compute $P(-2 < X < 10)$ where $X$ is the associated random variable.

\textit{Answer:}
If $a > 1$, then $F(0^+) < 0$, which would be invalid. If $a < 0$, then $F(x) > 1$ for $x \geq 0$, which is also invalid. For $\lim_{x \to \infty} F(x) = 1$, we need $b > 0$. Therefore $0 \leq a \leq 1$ and $b > 0$.

Since $F(x) = P(X < x)$:

\[P(-2 < X < 10) = P(X < 10) - P(X < -2)\]
\[= F(10) - F(-2)\]
\[= 1 - ae^{-10/b} - 0\]
\[= 1 - ae^{-10/b}\]
7. Preemptive Maintenance (20%)

Assume, for simplicity, that the lifetime of a machine is uniformly distributed in \([0, 1]\). If the machine fails, you face a cost equal to \(C\), which includes replacing the machine with a new one. Replacing the machine before it fails with a new one costs \(K < C\). You decide to replace the machine after \(T < 1\) time units or when it fails, whichever comes first.

(a) Let \(X\) be the random time when you replace the machine. Calculate \(E(X)\) in terms of \(T\).

**Answer:**

We replace the machine after either it fails or it reaches age \(T < 1\). Let \(W\) be the time when the machine fails. If \(W < T\), then we face a cost of \(C\) after \(W\) seconds, whereas if \(W > T\), we face a cost \(K\) after \(T\) seconds. Consequently, we face an average time equal to

\[
E(\min\{T, W\}) = \int_0^1 \min\{T, w\} dw = \int_0^T \int_1^T \int_0^1 Tdw = \frac{1}{2}T^2 + T(1 - T) = T - 0.5T^2.
\]

(b) Let \(Y\) be the random cost when you replace the machine (either \(K\) or \(C\)). Calculate \(E(Y)\) in terms of \(T\).

**Answer:**

We face a cost

\[
E[Y] = C \cdot P(W < T) + K \cdot P(W \geq T) = CT + K(1 - T)
\]

(c) The average replacement cost per unit of time is \(E(Y)/E(X)\). Find the value of \(T\) that minimizes that average cost.

**Answer:**

The average cost per unit of time is equal to

\[
\frac{CT + K(1 - T)}{T - 0.5T^2} = \frac{K + (C - K)T}{T - 0.5T^2}.
\]

To minimize that cost, we set the derivative with respect to \(T\) equal to zero. We find

\[
[K + (C - K)T](1 - T) = (T - 0.5T^2)(C - K),
\]

so that

\[
K + (C - 2K)T - (C - K)T^2 = (C - K)T - 0.5(C - K)T^2,
\]

i.e.,

\[
K - KT - 0.5(C - K)T^2 = 0,
\]

so that

\[
T = \frac{-K + \sqrt{K^2 + 2K(C - K)}}{C - K} = \frac{-K + \sqrt{2KC - K^2}}{C - K}.
\]