1. LLSE (5%) 
Let $X, Y$ be i.i.d. and uniformly distributed in $[-1, 1]$. Find $L[X|(X + Y)^2]$.

Answer. Let $Z = (X + Y)^2$. We know that 

$$L[X|Z] = E(X) + \frac{\text{cov}(X, Z)}{\text{var}(Z)}(Z - E(X)).$$

Now, 

$$\text{cov}(X, Z) = E(XZ) - E(X)E(Z) = E(XZ) = E(X(X^2 + 2XY + Y^2)) = 0.$$

Hence, 

$$L[X|(X + Y)^2] = 0.$$
2. LLSE (10%)
Let \( X, Y \) be i.i.d. \( N(0, 1) \). Find \( L[X \mid (X + Y)^3] \).

(Hint: Let \( V \) be a \( N(0, 1) \) random variable. Then \( E(V^4) = 3, E(V^6) = 15, E(V^8) = 105. \)

Answer. Let \( Z = (X + Y)^3 \). Then

\[
L[X \mid Z] = E(X) + \frac{\text{cov}(X, Z)}{\text{var}(Z)}(Z - E(Z)).
\]

Now,

\[
E(X) = E(Z) = 0
\]
\[
\text{cov}(X, Z) = E(XZ) - E(X)E(Z) = E(XZ) = E(X(X^3 + 3X^2Y + 3XY^2 + Y^3)) = 3 + 0 + 3 + 0 = 6
\]
\[
\text{var}(Z) = E(Z^2) = E((\sqrt{2}V)^6) = (\sqrt{2})^6 E(V^6) = 8 \times 15 = 120.
\]

Hence,

\[
L[X \mid Z] = \frac{6}{120} Z = \frac{1}{20} Z.
\]
3. **MMSE (5%)** Let $X, Y$ be i.i.d. $N(0, 1)$. Find $E[X|(X + Y)^3]$.

**Answer.** Let $Z = (X + Y)^3$. Given $Z$, one finds $X + Y = Z^{1/3}$. By symmetry, $E[X|X + Y] = (X + Y)/2$. Hence,

$$E[X|Z] = \frac{1}{2}Z^{1/3}.$$
4. MMSE (10%)
Let \(X, Y, Z\) be i.i.d. \(N(0,1)\). Calculate \(E[X|X + 3Y, X + 5Z]\).

Answer. Let \(V_1 = X + 3Y, V_2 = X + 5Z,\) and \(V = [V_1, V_2]^T\). Then, since \(X, V_1, V_2\) are zero-mean, 
\[
E[X|V] = \text{cov}(X, V)\text{cov}(V)^{-1}V.
\]
Now, 
\[
\text{cov}(X, V) = E(XV^T) = E(X[V_1, V_2]) = E([X^2 + 3XY, X^2 + 5XZ]) = [1, 1].
\]
Also, 
\[
\text{var}(V_1) = \text{var}(X) + \text{var}(3Y) = 1 + 9 = 10
\]
\[
\text{var}(V_2) = \text{var}(X) + \text{var}(5Z) = 1 + 25 = 26
\]
\[
\text{cov}(V_1, V_2) = E((X + 3Y)(X + 5Z)) = E(X^2 + 3XY + 5XZ + 15YZ) = 1,
\]
so that 
\[
\Sigma_V := \text{cov}(V) = \begin{bmatrix} 10 & 1 \\ 1 & 26 \end{bmatrix}.
\]
Hence, 
\[
E[X|V] = [1, 1] \begin{bmatrix} 10 & 1 \\ 1 & 26 \end{bmatrix}^{-1}V = [1, 1] \frac{1}{259} \begin{bmatrix} 26 & -1 \\ -1 & 10 \end{bmatrix}V = \frac{1}{259}[25, 9]V.
\]
5. MLE (10%)

For \( i \in \{1, 2, 3, 4\} \), if \( X = i \), then \( Y = N(\mu_i, \sigma^2 I) \) where \( \mu_1 = (-1, -1), \mu_2 = (1, -1), \mu_3 = (1, 1), \mu_4 = (-1, 1) \). Find the Maximum Likelihood Estimate of \( X \) given \( Y = y \in \mathbb{R}^2 \).

Answer. We saw in class that \( \text{MLE}[X|Y = y] = \arg\min_i ||y - \mu_i|| \) where the norm is the standard Euclidean norm. Accordingly,

\[
\text{MLE}[X|Y = y] = \begin{cases} 
1, & \text{if } y_1 < 0 \text{ and } y_2 < 0 \\
2, & \text{if } y_1 \geq 0 \text{ and } y_2 < 0 \\
3, & \text{if } y_1 \geq 0 \text{ and } y_2 \geq 0 \\
4, & \text{if } y_1 < 0 \text{ and } y_2 \geq 0.
\end{cases}
\]
6. **HT (10%)** If $X = 1$, then $Y$ is $N(1,1)$. If $X = 0$, then $Y$ is exponentially distributed with mean 1. Find the function $\hat{X}$ of $Y$ that maximizes $P[\hat{X} = 1|X = 1]$ subject to $P[\hat{X} = 1|X = 0] \leq 0.1$.

(*Hint: \(\cosh(x) = (e^x + e^{-x})/2, \sinh(x) = (e^x - e^{-x})/2\). You may give the answer in terms of a basic function without evaluating the numerical value.*)

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**Answer.** The likelihood ratio is

\[ L(y) = \frac{\phi_1(y)}{\phi_0(y)} = e^y \frac{1}{\sqrt{2\pi}} e^{-(y - 1)^2/2}. \]

Thus,

\[ L(y) = Ae^{-(y - 2)^2/2} \]

with $A = e^{-3/2}/\sqrt{2\pi}$.

Accordingly,

\[ \hat{X} = 1 \{ L(Y) > \lambda \} = 1 \{|y - 2| < \alpha \} \]

where $\alpha$ is such that

\[ P[|Y - 2| < \alpha |X = 0] = 0.1. \]

Now,

\[
P[|Y - 2| < \alpha |X = 0] = P[Y > 2 - \alpha |X = 0] - P[Y > 2 + \alpha |X = 0] = e^{-2 - \alpha} - e^{-2 + \alpha} = e^{-2} [e^{\alpha} - e^{-\alpha}] = 2e^{-2} \sinh(\alpha).
\]

Hence,

\[ \alpha = \sinh^{-1}(0.05e^2) \approx 0.36. \]

(The last value is found using a calculator.)

Finally,

\[ \hat{X} = 1 \{1.64 < Y < 2.36 \}. \]
7. CLT (10%) We want to estimate the probability $p$ that a coin flip yields ‘Tail.’ We flip the coin $n \gg 1$ times and we let $X_n$ denote the fraction of ‘Tails.’ Assume that we know that $p$ is not far from 0.5 but we want a more precise estimate. Using the Central Limit Theorem, find a value of $\alpha$ so that

$$P(X_n - \alpha < p < X_n + \alpha) \approx 0.95.$$

(Hint: If $V = N(0, 1)$, then $P(|V| < 1.96) \approx 0.95$.)

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**Answer.** We know, from the CLT, that

$$(X_n - p)\sqrt{n} \approx N(0, p(1 - p)) \approx N(0, \frac{1}{4}) = \frac{1}{2} V.$$

Thus,

$$X_n - p \approx \frac{1}{2\sqrt{n}} V.$$

Consequently,

$$P(|X_n - p| < \alpha) \approx P\left(\frac{1}{2\sqrt{n}} |V| < \alpha\right) = P(|V| < 2\sqrt{n} \alpha).$$

Accordingly, we need

$$2\sqrt{n} \alpha \approx 1.96, \text{ i.e.}, \alpha \approx \frac{1}{\sqrt{n}}.$$

Hence, for $n \gg 1$,

$$P\left(X_n - \frac{1}{\sqrt{n}} < p < X_n + \frac{1}{\sqrt{n}}\right) \approx 0.95.$$
8. Markov Chain (10%)

Consider a Markov $X_n$ on $\{1, 2, 3, 4\}$ such that

$$P(1, 2) = P(2, 3) = P(3, 4) = P(4, 1) = p \text{ and } P(2, 1) = P(3, 2) = P(4, 3) = P(1, 4) = 1 - p$$

where $p \in (0, 1)$ is given.

a) Is the Markov chain irreducible? Aperiodic?

b) Find the invariant distribution $\pi$.

c) Calculate the average time it takes the Markov chain to go from 1 to 3.

d) What is the probability that the Markov chain goes from 1 to 3 without first visiting 4?

Answer. a) The Markov chain is irreducible and periodic with period 2.

b) By symmetry, it must be that $\pi = [1/4, 1/4, 1/4, 1/4]$.

c) Let $\beta(i)$ be the mean time from $X_0 = i$ to 3. We find

$$\beta(1) = 1 + p\beta(2) + (1 - p)\beta(4)$$

$$\beta(2) = 1 + p \times 0 + (1 - p)\beta(1)$$

$$\beta(4) = 1 + p\beta(1) + (1 - p) \times 0.$$ 

Substituting the values for $\beta(2)$ and $\beta(4)$ found the the last two equations into the first one, we find

$$\beta(1) = 1 + p[1 + (1 - p)\beta(1)] + (1 - p)[1 + p\beta(1)].$$

Regrouping terms, we find

$$\beta(1) = 2[1 - 2p(1 - p)]^{-1},$$

which is the desired answer.

d) Let $\alpha(i)$ denote the probability that the Markov chain started in $X_0 = i$ visits state 3 before visiting state 4. We find

$$\alpha(1) = p\alpha(2) + (1 - p) \times 0$$

$$\alpha(2) = p + (1 - p)\alpha(1)$$

Substituting the expression for $\alpha(2)$ given by the last equation into the first one, we find

$$\alpha(1) = p[p + (1 - p)\alpha(1)] = p^2 + p(1 - p)\alpha(1).$$

That is,

$$\alpha(1) = p^2[1 - p(1 - p)]^{-1}.$$
9. **Conditional Expectation (10%)** Random variables $X$ and $Y$ are jointly distributed such that $X$ is uniformly distributed on $[0, 2]$ and, conditioned on $X$, $Y$ is then uniform on $[0, X]$. What is $E[X|Y]$?

*(Note: Be sure you solve for $E[X|Y]$, not $E[Y|X]$.)*

**Answer.** Start from the definition of conditional expectation:

$$E[X|Y = y] = \int_{-\infty}^{\infty} xf_{X|Y}(x|y)dx$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

We will solve for marginal distribution $f_Y(y)$ by way of the joint distribution $f_{XY}(x, y)$:

$$f_{XY}(x, y) = f_{Y|X}(y|x)f_X(x)$$

$$= \frac{1}{2}, \quad 0 \leq y \leq x, \quad 0 \leq x \leq 2$$

*Note that this triangle can also be described by $0 \leq y \leq 2, y \leq x \leq 2$.*

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

$$= \int_{y}^{2} \frac{1}{2x} dx$$

$$= \frac{1}{2} \ln x \bigg|_{x=y}^{2}$$

$$= \frac{\ln 2 - \ln y}{2}, \quad 0 \leq y \leq 2$$

$$f_{X|Y}(x|y) = \frac{1}{x \ln 2 - \ln y}, \quad 0 \leq y \leq 2, \quad y \leq x \leq 2$$

$$E[X|Y = y] = \int_{y}^{2} \frac{1}{x \ln 2 - \ln y} dx$$

$$= \frac{2 - y}{\ln 2 - \ln y}$$
10. MAP (10%) You are told that random variables $X$ and $Y$ are distributed as follows:

$$f_X(x) = \begin{cases} 
  ax^2, & 0 \leq x \leq 3 \\
  0, & \text{otherwise}
\end{cases}$$

$$f_{Y|X}(y|x) = xe^{-xy}, \quad 0 \leq x \leq 3, \quad 0 < y$$

where $a$ is some constant. Find $a$ and the MAP estimate for $X$ based on observation $Y$.

**Answer.** Since $f_X(x)$ is a probability distribution, its integral should equal 1.

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{0}^{3} ax^2 dx = a\left. \frac{x^3}{3} \right|_{x=0}^{3} = 9a = 1 \Rightarrow a = \frac{1}{9}$$

The MAP estimate is:

$$\hat{x} = \arg\max_x f_{X|Y}(x|y) = \arg\max_x \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} = \arg\max_x xe^{-xy}\frac{x^2}{9}$$

$$= \arg\max_{0\leq x \leq 3} e^{-xy}\frac{x^3}{9}$$

We maximize this by differentiating with respect to $x$ and setting equal to 0:

$$\frac{\partial}{\partial x} e^{-xy}\frac{x^3}{9} = -ye^{-xy}\frac{x^3}{9} + e^{-xy}\frac{x^2}{3}$$

$$= (-xy + 3)e^{-xy}\frac{x^2}{9} = 0$$

For $y > 0$, this has solutions $x = \frac{3}{y}$, $+\infty$, and 0. However, only for the case $x = \frac{3}{y}$ is $f_{X|Y}(x|y)$ positive (the other two are 0). Therefore, $\hat{x} = \frac{3}{y}$ if $0 \leq \frac{3}{y} \leq 3 \rightarrow y \geq 1$. For $0 < y < 1$, $\frac{3}{y} > 3$ so we know $f_{X|Y}(x|y)$ is increasing over $0 \leq x \leq 3$, and we choose $\hat{x} = 3$. Therefore the MAP estimator is:

$$\hat{x} = \begin{cases} 
  3, & 0 < y < 1 \\
  \frac{3}{y}, & 1 \leq y
\end{cases}$$
11. (10%). Note: This problem is quite difficult and its solution is not very short.

When $X = 1$, a machine produces electric light bulbs with i.i.d. exponential lifetimes with a mean less than one year. When $X = 0$, their mean lifetime is equal to one year. You measure the lifetimes $Y_1, \ldots, Y_n$ of $n \gg 1$ light bulbs. You decide $\hat{X} \in \{0, 1\}$ based on the observed lifetimes.

1) Construct $\hat{X}$ to maximize $P[\hat{X} = 1 | X = 1]$ subject to $P[\hat{X} = 1 | X = 0] \leq 5\%$.

2) Assume that when $X = 1$ the mean life time is 0.9 year. How large should $n$ be so that $P[\hat{X} = 1 | X = 1] \geq 95\%$?

(Hints: Use the CLT. Tables show that $P(N(0, 1) > 1.64) = 5\%$, $P(N(0, 1) > 1.96) = 2.5\%$, $P(N(0, 1) > 2.33) = 1\%$, $P(N(0, 1) > 2.53) = 0.5\%$. Also, the variance of an exponentially distributed random variable with rate $\mu$ is $1/\mu^2$.)

**Answer.** 1) For any given mean lifetime less than one year when $X = 1$, the likelihood ratio is decreasing in $Y_1 + \cdots + Y_n$. Consequently,

$$\hat{X} = 1 \{ \frac{Y_1 + \cdots + Y_n}{n} \leq A \}$$

where $A$ is such that

$$P[\frac{Y_1 + \cdots + Y_n}{n} \leq A | X = 0] = 5\%.$$ 

Now,

$$P[\frac{Y_1 + \cdots + Y_n}{n} \leq A | X = 0] = P(\frac{Z_1 + \cdots + Z_n}{n} \leq A)$$

where $Z_1, Z_2, \ldots$ are i.i.d. exponentially distributed with mean 1. Also,

$$P(\frac{Z_1 + \cdots + Z_n}{n} \leq A) = P(\frac{Z_1 + \cdots + Z_n}{n} - 1) \sqrt{n} \leq (A - 1) \sqrt{n}).$$

But, by the CLT,

$$\frac{Z_1 + \cdots + Z_n - 1) \sqrt{n}}{n} \approx N(0, \text{var}(Z_1)) = N(0, 1).$$

Hence,

$$P(\frac{Z_1 + \cdots + Z_n}{n} \leq A) \approx P(N(0, 1) \leq (A - 1) \sqrt{n}).$$

For this probability to equal to 5\%, we need

$$(A - 1) \sqrt{n} = -1.64, \text{ or } A = 1 - \frac{1.64}{\sqrt{n}}.$$ 

Hence,

$$\hat{X} = 1 \{ \frac{Y_1 + \cdots + Y_n}{n} \leq 1 - \frac{1.64}{\sqrt{n}} \}.$$ 

2) We find that

$$P[\hat{X} = 1 | X = 1] = P[\frac{Y_1 + \cdots + Y_n}{n} \leq A | X = 1] = P(\frac{V_1 + \cdots + V_n}{n} \leq A)$$
where the $V_m$ are i.i.d. exponentially distributed with mean 0.9 year. Now,

$$P\left(\frac{V_1 + \cdots + V_n}{n} \leq A\right) = P\left(\left[\frac{V_1 + \cdots + V_n}{n} - 0.9\right]\sqrt{n} \leq (A - 0.9)\sqrt{n}\right).$$

But, by the CLT,

$$\left[\frac{V_1 + \cdots + V_n}{n} - 0.9\right]\sqrt{n} \approx N(0, \text{var}(V_1)) = N(0, (0.9)^2).$$

Hence,

$$P[\hat{X} = 1|X = 1] \approx P(N(0, (0.9)^2) \leq (A - 0.9)\sqrt{n}) = P(N(0, 1) \leq \frac{(A - 0.9)\sqrt{n}}{0.9}).$$

Thus, for this probability to be 95%, we need

$$\frac{(A - 0.9)\sqrt{n}}{0.9} = 1.64,$$

so that

$$A = \frac{1.64 \times 0.9}{\sqrt{n}} + 0.9.$$

Combining with the expression for $A$ derived in part 1), we find

$$1 - \frac{1.64}{\sqrt{n}} = \frac{1.64 \times 0.9}{\sqrt{n}} + 0.9.$$

Solving for $n$ we find

$$n = \left[1.64 \times 0.1\right]^2 \approx 971.$$