EE 126 Fall 2007 Midterm #2
Thursday November 15, 3:30–5pm

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

- You have 90 minutes to complete the quiz.
- Write your solutions in the exam booklet. We will not consider any work not in the exam booklet.
- This quiz has three problems that are in no particular order of difficulty.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of numbers that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- This is a closed-book exam except for one single-sided, handwritten, 8.5×11 formula sheet plus a calculator.
- Be neat! If we can't read it, we can't grade it.
- At the end of the quiz, turn in your solutions along with this quiz (this piece of paper).

Problem	Score
1 [16 points]	
2 [13 points]	
3 [11 points]	
Total	

Problem 1: 16 points

A fly lays T eggs, where T has a Poisson distribution with parameter $\lambda > 0$. The weight of each egg follows an exponential distribution with parameter $\mu > 0$, independently of any other egg. Let W denote the total weight of the eggs that are laid.

(*Remember:* The Poisson with parameter λ has mean λ and variance λ; the exponential has mean $1/\mu$ and variance $1/\mu^2$.)

(a) (3 pts) Suppose that you observe that $T = t$ eggs are laid. Compute the mean and variance of W conditioned on $\{T = t\}$.

(b) (3 pts) Compute the expectation of W conditioned on $\{T \leq 3\}$.

(c) (3 pts) Compute the moment generating function of W. (For this part, nothing about T is observed.)

Now suppose that any egg with weight less than μ does not hatch, whereas any egg with weight greater than or equal to μ hatches with probability p, independently of any other egg.

(d) (4 pts) Compute the probability h that any given egg hatches. Letting F denote the total number of fly eggs that hatch, compute the expectation and variance of F, as well as the covariance $\text{cov}(F, T)$.

(e) (3 pts) Let V be the total weight of the eggs that hatched. Compute the expectation of V.

Solution 1:

(a) Conditioned on $T = t$, the total weight is given by the (non-random) sum $\sum_{i=1}^{t} V_i$, where the V_i are i.i.d., each exponential with parameter μ. By linearity of expectation and the i.i.d. nature, we have

$$\mathbb{E}[\sum_{i=1}^{t} V_i] \quad = \quad t\,\mathbb{E}[V_i] \; = \; t/\mu,$$

and by the i.i.d. nature, we have

$$\text{var}(\sum_{i=1}^{t} V_i) \quad = \quad t\,\text{var}(V_i) = t/\mu^2.$$

(b) By the conditional form of total expectation, we have

$$\begin{aligned}
\mathbb{E}[W \mid T \leq 3] &= \sum_{t=0}^{3} \mathbb{P}[T = t \mid T \leq 3]\mathbb{E}[W \mid T = t, T \leq 3] \\
&= \sum_{t=1}^{3} \frac{\mathbb{P}[T = t]}{\mathbb{P}[T \leq 3]} \, \mathbb{E}[W \mid T = t] \\
&= \sum_{t=1}^{3} \frac{\lambda^t \exp(-\lambda)/t!}{\sum_{k=0}^{3} \lambda^k \exp(-\lambda)/k!} \, \frac{t}{\mu} \\
&= \sum_{t=1}^{3} \frac{\lambda^t/t!}{\sum_{k=0}^{3} \lambda^k/k!} \, \frac{t}{\mu}.
\end{aligned}$$

(**Note:** Many students used the weights $\mathbb{P}[T = t]$ instead of the correct ones $\mathbb{P}[T = t \mid T \leq 3]$ in this problem.)

(c) By conditional expectation and i.i.d. nature of the $\{V_i\}$, we have

$$M_W(s) \;=\; \mathbb{E}[\mathbb{E}[\exp(sW) \mid T = t]] \;=\; \sum_{t=0}^{\infty} \mathbb{P}_T(T = t) \left(\mathbb{E}[\exp(sV_1)]\right)^t$$

Using the form of the MGF for exponential RVs, we have

$$\begin{aligned}
M_W(s) &= \sum_{t=0}^{\infty} \frac{\lambda^t \, \exp(-\lambda)}{t!} \left(\frac{\mu}{\mu - s}\right)^t \\
&= \sum_{t=0}^{\infty} \frac{\lambda^t \, \exp(-\lambda)}{t!} \left(\frac{\mu}{\mu - s}\right)^t \\
&= \exp\left(-\lambda + \frac{\lambda\mu}{\mu - s}\right),
\end{aligned}$$

valid for $s < \mu$.

(d) By conditioning, the probability h that any given egg hatches is

$$\begin{aligned}
h &= p\mathbb{P}[E_i \geq \mu] \\
&= p \int_{\mu}^{\infty} \mu \exp(-\mu x) dx \\
&= p \exp(-\mu^2).
\end{aligned}$$

Conditioned on $T = t$, the number of eggs that hatch is binomial with probability h, so that $\mathbb{E}[F|T] = hT$. Applying iterated expectation then yields

$$\mathbb{E}[F] = \mathbb{E}[\mathbb{E}[F|T]] = \mathbb{E}[hT] \;=\; \lambda h,$$

and

$$\text{var}(F) = h(1 - h)\lambda + h^2\lambda \;=\; h\lambda.$$

Finally, we have $\mathbb{E}[T]\mathbb{E}[F] = \lambda^2 h$, and

$$\mathbb{E}[TF] \;=\; \mathbb{E}[T\mathbb{E}[F \mid T]] \;=\; \mathbb{E}[hT^2] \;=\; h(\lambda + \lambda^2).$$

Thus $\text{cov}(F, T) = h(\lambda + \lambda^2) - \lambda^2 h = h\lambda$.

(e) Conditioned on $T = t$, the total weight V of hatched eggs is a sum of a binomial $\text{Bin}(t, h)$ number of random weights. Each random weight has the distribution of the random variable $\{V_i \mid V_i \geq \mu\}$; by the memoryless property of the exponential, this is the same as $\mu + V_i'$, where V_i' is an independent exponential. Again, by iterated expectation, we have

$$\begin{aligned}
\mathbb{E}[V] &= \mathbb{E}[\mathbb{E}[V \mid |T = t]] \\
&= \mathbb{E}[Th\mathbb{E}[\mu + V_i']] \\
&= h\left\{\mathbb{E}[T]\ \left(\mu + \mathbb{E}[V_i']\right)\right\} \\
&= h\left\{\lambda\mu + \lambda/\mu\right\}.
\end{aligned}$$

Problem 2: 13 points

An edge detector is applied in order to detect edges in an image. Conditioned on an edge being present at some position, the detector response is Gaussian with mean 0 and variance σ^2, whereas conditioned on no edge being present, the detector response is zero-mean Gaussian with variance 1. Any position in the image has a probability p of containing an edge.

(a) (3 pts) Compute the mean and variance of the detector response X.

(b) (3 pts) Compute the conditional probability of an edge being present given that $|X| > 10$. Your answer should be expressed in terms of p, σ, and the standard Gaussian CDF $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-t^2/2)dt$.

Now suppose that you observe a noise-corrupted version $Y = X + W$, where $W \sim N(0, 1)$ is Gaussian noise, independent of X.

(c) (3 pts) Compute the *linear* least-squares estimate (LLSE) of X based on Y, as well as the mean-squared error of this estimate.

(d) (4 pts) Compute the optimal least-squares estimator of X based on Y. What happens to this optimal estimator in comparison to the linear estimator from (c) as $p \to 1$?

Solution 2:

(a) We condition on the presence/absence of the edge, an event denoted by G. We have $\mathbb{E}[X] = p\mathbb{E}[X \mid G] + (1-p)\mathbb{E}[X|G^c] = 0$, and

$$\begin{aligned} \mathrm{var}(X^2) \;=\; \mathbb{E}[X^2] &= p\mathbb{E}[X^2 \mid G] + (1-p)\mathbb{E}[X^2 \mid G^c] \\ &= p\sigma^2 + (1-p). \end{aligned}$$

Note that X is *not* a Gaussian variable—rather, it is a mixture of two Gaussians with different variances.

(b) By Bayes rule, we have

$$\begin{aligned} \mathbb{P}[G \mid |X| \geq 10] &= \frac{p\,\mathbb{P}[|X| \geq 10 \mid G]}{p\mathbb{P}[|X| \geq 10 \mid G] + (1-p)\mathbb{P}[|X| \geq 10 \mid G^c]} \\ &= \frac{p\,\mathbb{P}[|Z| \geq 10/\sigma]}{p\mathbb{P}[|Z| \geq 10/\sigma] + (1-p)\mathbb{P}[|Z| \geq 1]} \end{aligned}$$

where $Z \sim N(0, 1)$ is a standard normal variate. Hence we have

$$\mathbb{P}[G \mid |X| \geq 10] \;=\; \frac{p\;(2\Phi(-10/\sigma))}{p\;(2\Phi(-10/\sigma)) + (1-p)\;(2\Phi(-10))}$$

where we applied symmetry to reduce $\mathbb{P}[|Z| \geq 10/\sigma] = 1 - (\Phi(10/\sigma) - \Phi(-10/\sigma)) = 2\Phi(-10/\sigma)$.

(c) We compute $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and

$$\text{var}(Y) = \text{var}(X) + \text{var}(W) \;=\; p\sigma^2 + (1-p) + 1 \;=\; p\sigma^2 + 2 - p.$$

Also, we have

$$\text{cov}(X,Y) \;=\; \text{cov}(X, X+W) \;=\; \text{var}(X) \;=\; p\sigma^2 + (1-p),$$

using the fact that $\text{cov}(X,W) = 0$ by independence of X and W. Thus, the LLSE of X based on $Y = y$ is

$$\widehat{X}(y) \;=\; \frac{p\sigma^2 + (1-p)}{p\sigma^2 + 2 - p} y.$$

The variance of the LLSE is given by

$$\begin{aligned}(1-\rho^2)\,\text{var}(X) \;&=\; \left(1 - \frac{\text{cov}^2(X,Y)}{\text{var}(X)\,\text{var}(Y)}\right)\text{var}(X) \\ &=\; p\sigma^2 + (1-p) - \frac{(p\sigma^2 + (1-p))^2}{p\sigma^2 + 2 - p}.\end{aligned}$$

(d) Note that (X,Y) are *not* jointly Gaussian, since X is not even marginally Gaussian (from (a), it is a mixture of Gaussians). The question is asking us compute the conditional expectation $\mathbb{E}[X|Y]$. In order to do so, we condition on the presence/absence of an edge, writing

$$\mathbb{E}[X \mid Y = y] \;=\; \mathbb{P}[G \mid Y = y]\mathbb{E}[X \mid Y = y, G] + \mathbb{P}[G^c \mid Y = y]\mathbb{E}[X \mid Y = y, G^c]. \quad (1)$$

The key to this decomposition is that conditioned on G, the pair $(X, Y|G)$ are jointly Gaussian, and similarly for conditioning on G^c. Therefore, from results on jointly Gaussian varaites, we know that $(X \mid Y = y, G)$ and $(X \mid Y = y, G^c)$ are Gaussian with means

$$\mathbb{E}[X \mid Y = y, G] = \frac{\sigma^2}{\sigma^2 + 1} y, \qquad \text{and} \qquad \mathbb{E}[X \mid Y = y, G^c] = \frac{1}{2} y.$$

It remains to compute the weights $\mathbb{P}[G \mid Y = y]$ in our expression (1) for the optimal estimator. By Bayes' rule, we have

$$\begin{aligned}\mathbb{P}[G \mid Y = y] \;&=\; \frac{p f_{Y|G}(y \mid G)}{p f_{Y|G}(y \mid G) + (1-p) f_{Y|G^c}(y \mid G^c)} \\ &=\; \frac{\frac{p}{\sqrt{2\pi\sigma^2}} \exp(-y^2/2\sigma^2)}{\frac{p}{\sqrt{2\pi\sigma^2}} \exp(-y^2/2\sigma^2) + \frac{1-p}{\sqrt{2\pi}} \exp(-y^2/2)}.\end{aligned}$$

and $\mathbb{P}[G^c|Y = y] = 1 - \mathbb{P}[G^c|Y = y]$. As $p \to 1$, this BLSE becomes equivalent to the LLSE from (c).

Problem 3: 11 pts

The random variables X and Y are independent, and each is uniformly distributed on the interval $[0, a]$ for some fixed $a > 0$.

(a) (6 pts) Compute the CDFs and PDFs of the new random variables:

(i) $U = \min\{X, Y\}$.

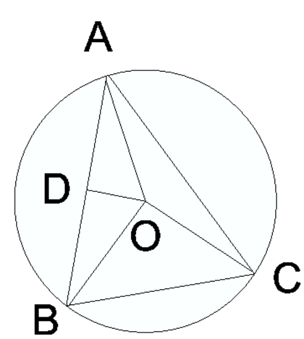
(ii) $W = a - \max\{X, Y\}$.

(b) (3 pts) A stick of length a is bent at a point X chosen uniformly along its length to form a right angle; the base of the resulting triangle has length X (uniform on $[0, a]$ as in part (a)). Suppose that this procedure is repeated independently for two different sticks. Compute the expected area of the triangle with the smallest base.
Hint: The area of a right triangle is equal to one half the base length times the height. Your answer to part (a) could be helpful here.

(c) (2 pts) Let O be the center of a circle of radius r, and consider the inscribed triangle $\triangle ABC$: it is completely specified by any two of the three angles $\angle AOB$, $\angle BOC$ and $\angle AOC$, as illustrated in the figure. (Given two angles, the third angle is specified since all three have to sum to 2π radians.)

Suppose that we generate a random triangle $\triangle ABC$ by choosing two angles independently and uniformly at random from the interval $[0, 2\pi]$. Let the random variable Θ_1 be the larger of the two angles and let Θ_2 the smaller of the two angles. Compute the joint distribution of (Θ_1, Θ_2). Use it to show that the expected area of triangle $\triangle ABC$ is $3r^2/(2\pi)$.

Hint: The area of subtriangle $\triangle AOB$ is $\frac{r^2}{2}\sin(\angle AOB)$, with similar formulas for the other subtriangles. Do not try this problem until you have completed the rest of the exam.



Solution 3:

(a) For $a \in [0, u]$, we have (using independence)

$$1 - F_U(u) \;=\; \mathbb{P}[X \geq u]\mathbb{P}[Y \geq u] \;=\; \frac{(a-u)^2}{a^2}.$$

Moreover $F_U(u) = 0$ for $u \leq 0$ and $F_U(u) = 1$ for $u \geq a$. and by differentiating, $f_U(u) = 2(a-u)/a^2$ for $u \in [0, a]$, and 0 otherwise.

Similarly, for $w \in [0, a]$, we have

$$F_W(w) \;=\; \mathbb{P}[r - \max\{X, Y\} \leq w] \;=\; 1 - \mathbb{P}[\max\{X, Y\} \leq a - w] \;=\; 1 - \frac{(a-w)^2}{a^2}.$$

Differentiating yields $f_W(w) = 2(a-w)/a^2$ for $w \in [0, a]$, and 0 otherwise.

(b) The area T of the specified triangle is given by $T = \frac{1}{2}U\,(a - U)$, where $U = \min\{X, Y\}$. Using the form of the density f_U computed in (a), we have

$$\begin{aligned}
\mathbb{E}[T] &= \int_0^a \frac{u(a-u)^2}{a^2} du \\
&= a^2 \left\{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right\} \\
&= \frac{a^2}{12},
\end{aligned}$$

by expanding the square, and performing the integration.

(c) Since the three angles must be non-negative and sum to 2π, we model them as $(\Theta_1, \Theta_2 - \Theta_1, 2\pi - \Theta_2)$ The joint PDF of Θ_1 and Θ_2 is

$$f_{\Theta_1,\Theta_2}(\theta_1, \theta_2) \;=\; \begin{cases} \frac{2}{(2\pi)^2} & \text{if } 0 \leq \theta_1 \leq \theta_2 \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Let $S_{\triangle ABC}$ denote the area of triangle ABC. We decompose this area into sums/differences of the areas of the subtriangles $\triangle AOC$, $\triangle AOB$ and $\triangle BOC$, associated with the angles Θ_1, $\Theta_2 - \Theta_1$ and $2\pi - \Theta_2$ respectively. We first claim from the hint on triangle areas, we always have $S_{\triangle ABC} = \frac{r^2}{2}\sin(\Theta_1) + \frac{r^2}{2}\sin(\Theta_2 - \Theta_1) - \frac{r^2}{2}\sin(\Theta_2)$. To see this, suppose first that all three of $(\Theta_1, \Theta_2 - \Theta_1, 2\pi - \Theta_2)$ lie within $[0, \pi]$ as shown in the diagram, then

$$\begin{aligned}
S_{\triangle ABC} &= S_{\triangle AOC} + S_{\triangle AOB} + S_{\triangle BOC} \\
&= \frac{r^2}{2}\sin(\Theta_1) + \frac{r^2}{2}\sin(\Theta_2 - \Theta_1) + \frac{r^2}{2}\sin(2\pi - \Theta_2) \\
&= \frac{r^2}{2}\sin(\Theta_1) + \frac{r^2}{2}\sin(\Theta_2 - \Theta_1) - \frac{r^2}{2}\sin(\Theta_2).
\end{aligned}$$

Otherwise, if $2\pi - \Theta_2 \geq \pi$, then we have

$$S_{\triangle ABC} \;=\; S_{\triangle AOC} + S_{\triangle AOB} - S_{\triangle BOC} \;=\; \frac{r^2}{2}\sin(\Theta_1) + \frac{r^2}{2}\sin(\Theta_2 - \Theta_1) - \frac{r^2}{2}\sin(\Theta_2)$$

The remaining cases (e.g., $\Theta_1 \leq \pi, \Theta_2 - \Theta_1 \geq \pi$) etc. are verified similarly. Thus, expected area of the triangle is

$$\frac{2}{(2\pi)^2} \times \frac{r^2}{2} \int_0^{2\pi} \left\{ \int_{\theta_1}^{2\pi} \left[\sin\theta_1 + \sin(\theta_2 - \theta_1) + - \sin(\theta_2) \right] d\theta_2 \right\} d\theta_1 = \frac{3r^2}{2\pi}.$$