Question 1. Let \( \{X, Y, Z\} \) be independent \( N(0, 1) \) random variables.

a. (14%) Calculate \( E[3X + 5Y \mid 2X - Y, X + Z] \).

b. (6%) How does the expression change if \( X, Y, Z \) are i.i.d. \( N(1, 1) \)?

a. Let \( V_1 = 2X - Y, V_2 = X + Z \) and \( V = [V_1, V_2]^T \). Then

\[
E[3X + 5Y \mid V] = a \Sigma_V^{-1} V
\]

where

\[
a = E((3X + 5Y)V^T) = [1, 3]
\]

and

\[
\Sigma_V = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}.
\]

Hence,

\[
E[3X + 5Y \mid V] = [1, 3] \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1} V = [1, 3] \frac{1}{6} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} V
\]

\[
= \frac{1}{6}[-4, 13] V = -\frac{2}{3} (2X - Y) + \frac{13}{6} (X + Z).
\]

b. Now,

\[
E[3X + 5Y \mid V] = E(3X + 5Y) + a \Sigma_V^{-1} (V - E(V)) = 8 + \frac{1}{6}[-4, 13] (V - [1, 2]^T)
\]

\[
= \frac{26}{6} - \frac{2}{3} (2X - Y) + \frac{13}{6} (X + Z).
\]
**Question 2.** 25%. Let $X, Y$ be independent random variables uniformly distributed in $[0, 1]$. Calculate $L[Y^2 \mid 2X + Y]$.

One has

$$L[Y^2 \mid 2X + Y] = E(X^2) + \frac{E(Y^2(2X + Y)) - E(Y^2)E(2X + Y)}{\text{var}(2X + Y)}(2X + Y - E(2X + Y))$$

$$= \frac{1}{3} + \frac{1/3 + 1/4 - (1/3)(3/2)}{4(1/3 - 1/4) + (1/3 - 1/4)}(2X + Y - 3/2).$$
**Question 3.** 15\%. Let \( \{X_n, n \geq 1\} \) be independent \( N(0, 1) \) random variables. Define \( Y_{n+1} = aY_n + (1 - a)X_{n+1} \) for \( n \geq 0 \) where \( Y_0 \) is a \( N(0, \sigma^2) \) random variable independent of \( \{X_n, n \geq 0\} \). Calculate

\[
E[Y_{n+m} | Y_0, Y_1, \ldots, Y_n]
\]

for \( m, n \geq 0 \).

*Hint:* First argue that observing \( \{Y_0, Y_1, \ldots, Y_n\} \) is the same as observing \( \{Y_0, X_1, \ldots, X_n\} \). Second, get an expression for \( Y_{n+m} \) in terms of \( Y_0, X_1, \ldots, X_{n+m} \). Finally, use the independence of the basic random variables.

One has

\[
\begin{align*}
Y_{n+1} &= aY_n + (1 - a)X_{n+1}; \\
Y_{n+2} &= aY_{n+1} + (1 - a)X_{n+2} = a^2Y_n + (1 - a)X_{n+2} + (1 - a)^2X_{n+1}; \\
&\quad \vdots \\
Y_{n+m} &= a^mY_n + (1 - a)X_{n+m} + (1 - a)^2X_{n+m-1} + \cdots + (1 - a)^mX_{n+1}.
\end{align*}
\]

Hence,

\[
E[Y_{n+m} | Y_0, Y_1, \ldots, Y_n] = a^mY_n.
\]
**Question 4.** 20%. Given $\theta$, the random variables $\{X_n, n \geq 1\}$ are i.i.d. $U[0, \theta]$. Assume that $\theta$ is exponentially distributed with rate $\lambda$.

a. Find the MAP $\hat{\theta}_n$ of $\theta$ given $\{X_1, \ldots, X_n\}$.

b. Calculate $E(|\theta - \hat{\theta}_n|)$.

One finds that
\[
f[x \mid \theta]f(\theta) = \frac{1}{\theta^n}1\{x_k \leq \theta, k = 1, \ldots, n\} \lambda e^{-\lambda \theta}.
\]

Hence,
\[
\hat{\theta}_n = \max\{X_1, \ldots, X_n\}.
\]

Consequently, by symmetry,
\[
E[\theta - \hat{\theta}_n|\theta] = \frac{1}{n+1} \theta.
\]

Finally,
\[
E(|\theta - \hat{\theta}_n|) = E(E[|\theta - \hat{\theta}_n|\theta])) = \frac{1}{\lambda(n+1)}.
\]

A few words about the symmetry argument. Consider a circle with a circumference length equal to 1. Place $n + 1$ point independently and uniformly on that circumference. By symmetry, the average distance between two points is $1/(n+1)$. Pick any one point and open the circle at that point, calling one end 0 and the other end 1. The other $n$ points are distributed independently and uniformly on $[0, 1]$. So, the average distance between 1 and the closest point in $1/(n+1)$. Of course, we could do a direct calculation.
Question 5. 20%. Let $(X, Y)$ be jointly Gaussian. Show that $X - E[X \mid Y]$ is Gaussian and calculate its mean and variance.

We know that

$$E[X \mid Y] = E(X) + \frac{\text{cov}(X, Y)}{\text{var}(Y)}(Y - E(Y)).$$

Consequently,

$$X - E[X \mid Y] = X - E(X) - \frac{\text{cov}(X, Y)}{\text{var}(Y)}(Y - E(Y))$$

and is certainly Gaussian. This difference is zero-mean. Its variance is

$$\text{var}(X) + \left[\frac{\text{cov}(X, Y)}{\text{var}(Y)}\right]^2\text{var}(Y) - 2\frac{\text{cov}(X, Y)}{\text{var}(Y)}\text{cov}(X, Y) = \text{var}(X) - \frac{[\text{cov}(X, Y)]^2}{\text{var}(Y)}.$$