EECS 126: Probability and Random Processes

Midterm 2

Tuesday, November 12, 2002 3:40-5 pm

Answer all questions on the attached blank sheets. Please explain your reasoning carefully. Answers without reasoning will get no marks. The total is 100 points. Q. 2 (d) is a bonus part worth 15 points.

[20] 1. Let X and Y be independent RV's, both exponentially distributed with mean 1. Let $Z = \max(X, Y)$.

[10] a) Compute the pdf of X conditional on the event that $Z \leq z$.

[10] b) X and Y are independent, but conditional on $Z = z$, are they still independent? Explain. (Hint: no calculations needed.)

[35] 2.

[10] a) Suppose you have access to a RV $X \sim N(\mu_1, \sigma_1^2)$. Explain how you can use X to generate a RV Y which is $N(\mu_2, \sigma_2^2)$.

[10] b) Suppose you have access to a RV X uniform distributed in $[0, 1]$. Explain how you can use X to generate a continuous RV Y with a given pdf f_Y which is nonzero everywhere.

[15] c) Suppose you have access to a RV X uniform distributed in $[0, 1]$. Explain how you can use X to generate a *discrete* RV Y with a given pmf $p_Y(\cdot)$. You can assume that the range of values Y can take on is finite. (Hint: try the simple case when Y is Bernouilli (i.e.Y takes on only two possible values) and then attack the general case.)

[15] d) (Bonus) Suppose you have access to two indpendent RVs X and Y both uniform distributed in $[0, 1]$. Explain how you can use them to generate two continuous RV's U and V with a given joint density $f_{U,V}$ which is nonzero everywhere. (Hint: you may want to consider a sequential procedure where you first generate U and then generate V.)

[45] 3.

a) A continuous random variable U has a pdf of the form:

$$f_U(u) = a \exp(bu^2 + cu), \qquad -\infty < u < \infty.$$

where a, b, c are constants. a and b are non-zero.

[10] i) Can you say anything definitive about the signs of a, b and c?

[10] ii) Does U have to be Gaussian? If so, express its mean and variance in terms of a, b and c in the simplest way. If not, give an example of U with pdf of the above form but is not Gaussian.

[5] b) Let $X \sim N(0, v^2)$, $Z \sim N(0, \sigma^2)$ and we have a noisy observation:

$$Y = X + Z$$

The RVs X and Z are independent.

Find from first principles the MMSE estimate of X given Y and the resulting minimum mean square error. (Hint: you may find your answer to part (a)(ii) useful in simplifying the calculations.)

c) Suppose we now have n noisy observations:

$$Y_i = X + Z_i, \qquad i = 1, 2 \ldots n,$$

where the Z_i's are i.i.d. $N(0, \sigma^2)$ noise and independent of X.

[14] i) Find from first principles the MMSE estimate of X given $Y_1, \ldots, Y_n$ and the resulting minimum mean square error. (Hint: you may find your answer to part (a)(ii) VERY useful in simplifying the calculations.)

[3] ii) Explain intuitively the effect of the parameter v^2 on the MMSE estimator. How does the relative role of v^2 change when n becomes large?

[3] iii) What happens to the minimum mean-square error when n is large? Give an intuitive explanation.