• If we can't read it, we can't grade it.

• We can only give partial credit if you write out your derivations and reasoning in detail.

• You may use the back of the pages of the exam if you need more space.

*** GOOD LUCK! ***

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Partial Credit.

Partial credit will be given only if there is sufficient information in your work. In general, a good way to show that you understand what's going on is for example to provide plots, sketches, and formulas for intermediate signals. That way, if you make an error somewhere along the way, we can trace it and evaluate whether or not you understood the basics of the problem.

Useful Formulae.

- For the continuous-time box function,

\[
b(t) = \begin{cases} 
1, & -T \leq t \leq T \\
0, & \text{otherwise}
\end{cases}
\]

the (continuous-time) Fourier transform is given by

\[
B(j\Omega) = \frac{2\sin(\Omega T)}{\Omega}.
\]

- \(\tan(\pi/4) = 1\)
Problem 1 (Phase Properties.)

Given the phase characteristics of a generalized linear phase FIR filter \( H_a(e^{j\omega}) \) shown below, answer the following questions. Include brief explanations to get credit.

\[
\frac{H_a(e^{j\omega})}{\frac{\pi}{2}}
\]

(a) (6 pts) Is this a symmetric (i.e., Type-I or Type-II) or an anti-symmetric (i.e., Type-III or Type-IV) filter? Why?

\[
\text{Slope } = \frac{-\pi}{\frac{\pi}{2}} = -1 \rightarrow \alpha = 1 \rightarrow m = 2 \quad \text{(even)}
\]

\[
B = \frac{\pi}{2} \quad \text{and } m \text{ even } \rightarrow \quad \boxed{\text{Type-III (anti-symmetric) filter}}
\]

(b) (8 pts) Can the filter length be determined from the given information? If yes, what is the length? If not, why not?

\[
m = 2\alpha = 2 \cdot 1 = 2
\]

\[
\text{length } = m + 1 = 3
\]

(c) (6 pts) The filter magnitude response has a DC gain of 1. True or False? Why?

\[
|H(0)| = 1 \Rightarrow |h(0) + h(1) + h(2)| = |h(0) + (-h(0))| = 0
\]

but

\[
H(0) = h(0) + h(1) + h(2)
\]

\[
0 \neq 1. \quad \boxed{\text{False}}
\]
Problem 2 (Filter Design.)

(a) (12 pts) A continuous-time filter is given by \( H_a(s) = \frac{2}{s^2 + s + 1} \). We want to use this as a prototype filter to design a discrete-time filter via the bilinear transform with a suppression of \( 1/\sqrt{2} \) at \( \omega_c = \pi/2 \). (Note that the filter has a gain of 1 at frequency zero.) Give \( H(z) \) explicitly.

\[
H_a(s) = \frac{2}{s^2 + s + 1} \\
\omega_c = \frac{\pi}{2} \quad \Rightarrow \quad z = e^{\frac{i \pi}{2}}
\]

\[
H_a(z) = H_a(s) \bigg|_{s = \frac{2}{T_a} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)} = \frac{2}{2 + \frac{2}{T_a} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)}
\]

\[
H_a(z) = \frac{T_a (1+z^{-1})}{T_a (1+z^{-1}) + (1-z^{-1})} \quad \Rightarrow \quad H_a(z) = \frac{1}{\sqrt{2}} \frac{T_a (1-z^{-1})}{T_a (1-z^{-1}) + (1+z^{-1})}
\]

\[
|H_a(i)| = \frac{T_a \sqrt{2}}{\sqrt{T_a^2 + 1}} = \frac{1}{\sqrt{2}}
\]

\[
H_a(i) = \frac{T_a - T_a i}{(T_a + 1) + (1-Ta) i}
\]

\[
T_a^2 = 1 \quad \Rightarrow \quad T_a = 1
\]

\[
|H_a(z)| = \frac{1+z^{-1}}{2}
\]

(b) (3 pts) Sketch \( |H(e^{j\omega})| \) for the filter designed in part (a) over the interval \( 0 \leq \omega \leq \pi \).

\[
H_a(z) = \frac{z^{-1} + 1}{2z} \quad \Rightarrow \quad \frac{e^{j\omega} + 1}{2e^{j\omega}} \quad \Rightarrow \quad |H(e^{j\omega})| = \frac{|1+e^{j\omega}|}{2}
\]
Problem 3 (Filter Design.)

(a) (5 pts) We want to approximate the lowpass filter in Figure 1 with the optimal minimax (Parks-McClellan) Type-I filter \( h(n) \). Just like in class, the band \( 0 \leq \omega \leq \omega_p \) is the desired passband, and the band \( \omega_s \leq \omega \leq \pi \) is the desired stopband. In Figure 1, provide a sketch the form of \( H(e^{j\omega}) \) when \( h(n) \) has length 3. Recall that the amplitude of a Type-I filter has the form

\[
A(e^{j\omega}) = \sum_{k=0}^{M/2} a_k \cos(k\omega).
\]

Length 3 \( \Rightarrow M = 2 \) \( \Rightarrow 0 \) alternations

Assume that \( \omega_s = \pi - \omega_p \)

(b) (10 pts) Determine the filter \( h(n) \). Hint: If you find it easier, you may start by assuming that \( \omega_p = \pi/3 \) and thus, \( \omega_s = 2\pi/3 \).

\[
h(0) = h(2) \quad \text{and} \quad h(1) = 0
\]

Type-II Symmetry

\[
P - M \quad \text{guess:} \quad \omega_1 = 0 \quad \omega_2 = \omega_p = \frac{\pi}{3} \quad \omega_3 = \omega_s = \frac{2\pi}{3}
\]

\[
\begin{pmatrix}
1 & \cos(\omega_1) - K & a_0 \\
1 & \cos(\omega_2) & K \\
1 & \cos(\omega_3) & -1
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\delta
\end{pmatrix}
= \begin{pmatrix}
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -K & a_0 \\
1 & K & a_1 \\
1 & -1 & \delta
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
\delta
\end{pmatrix}
= \begin{pmatrix}
1
\end{pmatrix}
\]

\[
a_{0} + K \delta = 1
\]

Try new \( \omega_s \)'s, compare until minimax is achieved.

Parks-McClellan
(c) (5 pts) What is the largest value of $\omega_p$ such that the maximum error is $\delta \leq 1/6$?

$$\omega_p = \frac{\pi}{2}$$

$$E(\omega) = W(\omega) [H(\epsilon e^{i\omega}) - A \epsilon e^{i\omega}]$$
Problem 4 (Multirate System.)

\[ x[n] \rightarrow H_d \rightarrow y[n] \rightarrow 8 \uparrow \rightarrow H_i \rightarrow w[n] \rightarrow \text{ZOH} \rightarrow \text{Ha} \rightarrow z(t) \rightarrow y(t) \]

\[ H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{8} \\ 0, & \text{else} \end{cases} \quad H_i(e^{j\omega}) = \begin{cases} 8, & |\omega| \leq \frac{\pi}{8} \\ 0, & \text{else} \end{cases} \]

The ZOH operates at interval \( T \) but produces pulses of width \( \frac{T}{4} \), i.e.

\[ g(t) = \begin{cases} 1, & 0 \leq t \leq \frac{T}{4} \\ 0, & \text{else} \end{cases} \]

and the output is \( z(t) = \sum_{n=-\infty}^{\infty} w[n]g(t-nT) \).

(a) (9 pts) For \( X(e^{j\omega}) \) pictured, sketch \( W(e^{j\omega}) \). Label the magnitude and bandwidth.

(b) (7 pts) For the same \( X(e^{j\omega}) \), sketch \( |Z(j\Omega)| \) for \( \Omega = [\frac{-\pi}{T}, \frac{\pi}{T}] \).

\[ Z(j\Omega) = \sum_{n=-\infty}^{\infty} W(j\Omega) \xi(n) \]

\[ Z(j\Omega) = \sum_{n=-\infty}^{\infty} W(j\Omega - \frac{2\pi n}{T}) \]

\[ Z(j\Omega) = \sum_{n=-\infty}^{\infty} \Xi(j\Omega) \]

\[ Z(j\Omega) = \frac{1}{\Xi(j\Omega)} \]

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(c) (7 pts) For this part, we are interested in making the dashed block (with input $y[n]$ and output $y(t)$) an ideal D/A converter for arbitrary $y[n]$, i.e. ignore the effects of $H_d$.

What is the "cheapest" \footnote{the filter having the largest transition band (as we have seen in class, the smaller (i.e., steeper) the transition band, the more filter coefficients are necessary)} filter $H_o(j\Omega)$ such that between $y[n]$ and $y(t)$ we have an ideal D/A. Sketch $|H_o(j\Omega)|$ and specify its value where necessary.

\[
Z(t) = \sum_{n=-\infty}^{\infty} w(n) g(t-nT)
\]

\[
|H_o(j\Omega)|
\]

(d) (2 pts) Explain (in words) the advantages and disadvantages of this D/A converter design over a direct implementation (as we have discussed it in class).

\textbf{This D/A converter design is cheaper than the direct implementation but it is not as exact (due to the need to arrive at a cheap solution).}\n
\textbf{The ZOH approximation will not produce the same results as higher-order holds on the sinc kernel reconstruction.}
Problem 5 (Filter Bank.)

For the filter bank in Figure 2, find the conditions for perfect reconstruction, i.e., the conditions on the filters $F_i(z), G_i(z), H_i(z)$ (for $i = 0, 1$) such that $y[n] = x[n]$.

- $H_1(z) \sim \text{gain 1, \, \text{cutoff } \pi/2}$
- $H_0(z) \sim \text{gain 1, \, \text{cutoff } \pi/2}$
- $G_1(z) \sim \text{gain 2, \, \text{cutoff } \pi/2}$
- $G_0(z) \sim \text{gain 2, \, \text{cutoff } \pi/2}$

$F_1(z), F_0(z)$ must ensure that the two branches add up to $y[n]$, i.e., each branch is $y[n]/2$. (by symmetry)

$\frac{y[n]}{2} = \frac{x[n]}{2} \rightarrow F(z) = \frac{1}{2}$

$F_1(z) \sim \text{gain } \frac{1}{2}, \, \text{\text{cutoff } } \pi$  

$F_0(z) \sim \text{gain } \frac{1}{2}, \, \text{\text{cutoff } } \pi$

Keep all frequencies scale the input uniformly.

NOT JURE  WHAT YOU ARE DOING...