EECS 123 Digital Signal Processing
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University of California, Berkeley: Fall 2007
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Exam 1

- If we can't read it, we can't grade it.
- We can only give partial credit if you write out your derivations and reasoning in detail.
- You may use the back of the pages of the exam if you need more space.

*** GOOD LUCK! ***

Problem	Points earned	out of
Problem 1	9	18
Problem 2	30	30
Problem 3	6	30
Extra Credit	—	5
Problem 4	25	25
Total		103

0/5

Problem 1 *(Systems.)* 18 Points

(a) (5 Pts) For a certain signal $g[n]$, we define

$$h[n] = \begin{cases} g[n/64], & \text{if } ((n))_{64} = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Express $H(z)$ in terms of $G(z)$.

$$\boxed{H(z) = 64\,G(64z)}$$

time-frequency modulation

2/5

(b) (5 Pts) For a certain system, the following input/output relations hold for the system in question:

$$\delta[n] \longrightarrow 3\delta[n-2] + 4\delta[n-4] \tag{2}$$

$$2\delta[n] \longrightarrow 6\delta[n-2] + 8\delta[n-4] \tag{3}$$

$$\delta[n] + \delta[n-1] \longrightarrow 3\delta[n-2] + 3\delta[n-3] + 4\delta[n-4] + 4\delta[n-5] \tag{4}$$

We would like to know whether the system is linear, causal, shift-invariant, and BIBO stable. Indicate "**Y**" for Yes, "**N**" for No, and "**X**" for "cannot be determined due to insufficient information." *In order to discourage guessing, for a) and b) below only, each incorrect answer will be penalized 1 point (but don't worry, you can't get less than zero for this problem.)*

_Y_ Linear _N_ Causal _Y_ Shift-Invariant _X_ Stable

OPI ≈ output precedes input

$(L \wedge OPI) \Longrightarrow \neg C$

7/8

(c) (8 Pts) Consider the causal, linear, time-invariant discrete-time system that obeys the following difference equation:

$$y[n] - ay[n-2] = x[n] - bx[n-1] \tag{5}$$

- Find the transfer function $H(z)$.
- Find the impulse response $h[n]$.

$$\boxed{H(z) = \frac{1 - bz^{-1}}{1 - az^{-2}}}$$ ✓

$y(n) = x(n) - bx(n-1) + a[x(n-2) - bx(n-3) + \cdots$

$h(n) = \delta(n) - b\delta(n-1) + a[\delta(n-2) - b\,\delta(n-3) + a[\delta(n-4) - b\,\delta(n-5) + \cdots$

$h(n) = \delta(n) + a\delta(n-2) + a^2\delta(n-4) \ldots$
$\quad + b\,\delta(n-1) - ab\,\delta(n-3) - a^2 b\,\delta(n-5) \ldots$

$$\boxed{h(n) = \sum_{k=0}^{\infty} \left[a^k \delta(n-2k) - a^k b\,\delta(n-1-2k)\right]}$$

close, but need $a^{\frac{k}{2}}$, $a^{\frac{k-1}{2}}$

Problem 2 *(DTFT and DFT.)* 30 Points

Consider the signals

$$x_1[n] \;=\; \begin{cases} 1, & \text{for } -1 \le n \le 1, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

and

$$x_2[n] \;=\; \begin{cases} 3-|n|, & \text{for } -3 \le n \le 3, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Define

$$y[n] \;=\; x_1[n] * x_2[n] \tag{8}$$

$$z_N[n] \;=\; x_1[n] \circ_N x_2[n], \tag{9}$$

where the first is the linear convolution and the second is the N-point circular convolution.

10/10

(a) (10 Points) Find an explicit formula for $Y(e^{j\omega})$, the DTFT of $y[n]$. *Remark:* "Explicit formulas" cannot contain any unresolved sums or integrals.

$$y(n) = (x_1 * x_2)(n) \longleftrightarrow Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

$$X_1(e^{j\omega}) = e^{j\omega} + 1 + e^{-j\omega} = 1 + 2\cos\omega$$

$$X_2(e^{j\omega}) = e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega} = 3 + 4\cos\omega + 2\cos 2\omega$$

$$Y(e^{j\omega}) = [1+2\cos\omega][3+4\cos\omega+2\cos 2\omega]$$

$$= 3 + 4\cos\omega + 2\cos 2\omega + 6\cos\omega + 8\cos^2\omega + 4\cos\omega\cos 2\omega$$

$$= 3 + 10\cos\omega + 8\cos^2\omega + [2[2\cos^2\omega - 1]](1+2\cos\omega)$$

$$= 3 + 10\cos\omega + 8\cos^2\omega + 4\cos^2\omega + 8\cos^3\omega - 2 - 4\cos\omega$$

$$\boxed{Y(e^{j\omega}) = 1 + 6\cos\omega + 12\cos^2\omega + 8\cos^3\omega}$$

8/8

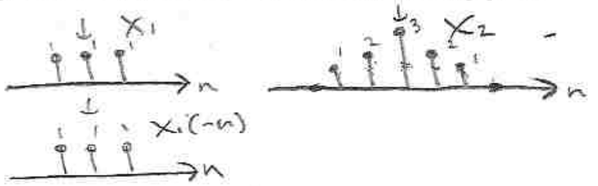
(b) (8 Points) Find an explicit formula for $Z_N[k]$, the N-point DFT of the signal $z_N[n]$, for all $N \ge 9$.

$N \geq L+P-1 \Longrightarrow$ circ conv = lin conv $\rightarrow y(n) = z_N(n) \Longrightarrow Z_N(k)$ fully samples $Y(e^{j\omega})$

$$\boxed{Z_N[k] = Y(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}} = 1 + 6\cos\left(\tfrac{2\pi k}{N}\right) + 12\cos^2\left(\tfrac{2\pi k}{N}\right) + 8\cos^3\left(\tfrac{2\pi k}{N}\right)$$

8/8

(c) (8 Points) Find an explicit formula for $Z_N[k]$, the N-point DFT of the signal $z_N[n]$, for $N = 5$.



$$Z_5(n) = (x_1 \circledS x_2)(n)$$

$$Z_5(n) = \begin{cases} 7 & n=0 \\ 6 & n=1 \\ 4 & n=2 \\ 4 & n=3 \\ 6 & n=4 \end{cases}$$

$$Z_5(k) = \sum_{n=0}^{4} x(n) W_5^{kn} = \boxed{7 + 6W_5^{k} + 4W_5^{2k} + 4W_5^{3k} + 6W_5^{4k}}$$

✓

$Z_N(k) = 7 + 6$

4/4

(d) (4 Points) Find an explicit formula for $Z_N[k]$, the N-point DFT of the signal $z_N[n]$, for $N = 7$.

$$Z_7(n) = \begin{cases} 7 & n=0 \\ 6 & n=1 \\ 3 & n=2 \\ 1 & n=3 \\ 1 & n=4 \\ 3 & n=5 \\ 6 & n=6 \end{cases}$$

$$Z_7(k) = \sum_{n=0}^{6} x(n) W_7^{kn} = \boxed{7 + 6W_7^{k} + 3W_7^{2k} + W_7^{3k} + W_7^{4k} + 3W_7^{5k} + 6W_7^{6k}}$$

✓

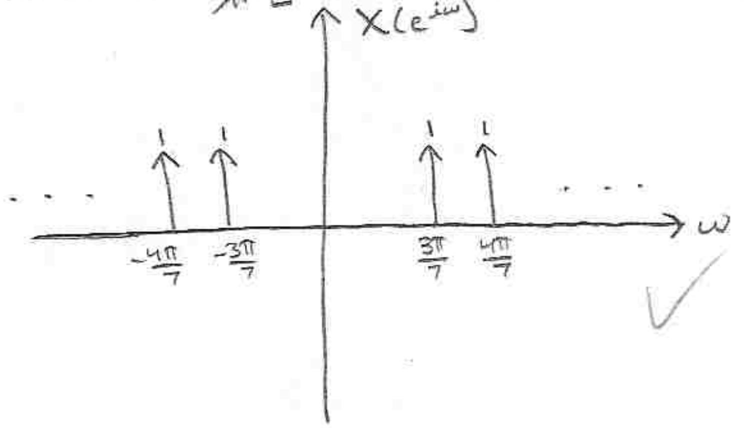
Problem 3 *(Spectral Analysis.)* 30 Points

(a) (3 Points) Sketch the DTFT of the signal

$$x[n] \;=\; \frac{1}{\pi}\cos\left(\frac{3\pi}{7}n\right)+\frac{1}{\pi}\cos\left(\frac{4\pi}{7}n\right) \tag{10}$$

Carefully label *both* axes.

$$X(e^{i\omega}) = \frac{1}{\cancel{\pi}}\left[\cancel{\pi}\left[\delta(\omega-\tfrac{3\pi}{7})+\delta(\omega+\tfrac{3\pi}{7})+\delta(\omega-\tfrac{4\pi}{7})+\delta(\omega-\tfrac{4\pi}{7})\right]\right]$$



3/3

(b) (10 Points) Now, suppose that our spectrum analyzer keeps $M+1$ samples, according to

$$y[n] \;=\; \begin{cases} x[n], & \text{for } -\frac{M}{2}\le n\le\frac{M}{2}, \\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

(Assume that M is an even integer.) Sketch the DTFT of the signal $y[n]$, again carefully labeling both axes, and marking the height of the main peaks.

$y(n) = x(n)\cdot w(n)$ $\qquad w(n) = \begin{cases} 1 & -\frac{M}{2}\le n\le\frac{M}{2} \\ 0 & e/w\end{cases}$

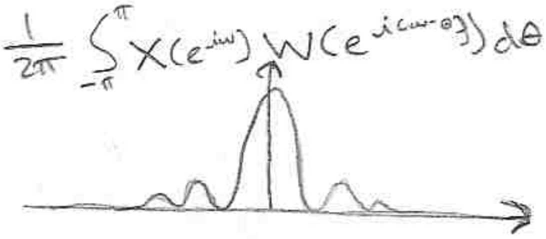
$$W(e^{i\omega}) = \sum_{n=-\frac{M}{2}}^{M/2} e^{-i\omega n} = \frac{\sin[\omega(M+1)/2]}{\sin[\omega/2]}$$

2/10

$$Y(e^{i\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})\,W(e^{i(\omega-\theta)})\,d\theta$$

$Y(e^{i\omega}) \approx X(e^{i\omega})$

$W(e^{i\omega})$



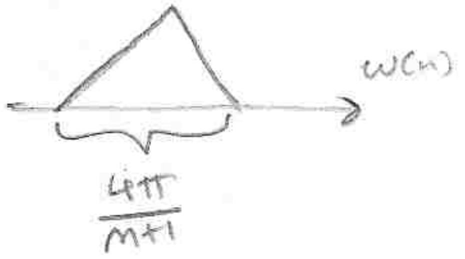
Also...

$$Y(e^{i\omega}) = \frac{1}{\pi}\sum_{n=-M/2}^{M/2}\left[\cos(\tfrac{3\pi}{7}n)+\cos(\tfrac{4\pi}{7}n)\right]e^{-i\omega n}$$

ok

So, what does it look like?

(c) (10 Points) We would like to find a suitable M that permits to resolve the distinct peaks in the spectrum of $x[n]$. To make things simple, let us use a simple straight-line approximation to the sinc-function: Simply connect the first zero crossing left and right to the main peak *by straight lines,* and set the rest of the sinc-function to zero. (In other words, inscribe the main lobe of the sinc by a triangle, and set the rest to zero.) For this simplified model, determine the conditions on M such that distinct peaks show up in $Y(e^{j\omega})$.



$$\frac{\cancel{4\pi}}{M+1} \geq 2\left(\frac{\cancel{4\pi}}{7}\right) \Rightarrow M+1 \leq \frac{7}{2}$$

$$M \leq \frac{7}{2} - 1$$

$$\boxed{M \leq \frac{5}{2}}$$

M SHOULD BE LARGE, NOT SMALL!

$\frac{0}{10}$

$\frac{0}{5}$

(c) (Extra Credit 5 Points)* Replace the straight-line approximation by a more precise approximation to obtain a more refined estimate of the suitable values of M. (Use the back of the page)

Use Hanning... $\boxed{w(n) = \begin{cases} 0.5 - 0.5\cos(2\pi n/M) & 0 \leq n \leq M \\ 0 & e/w \end{cases}}$

This will taper the edges and counter Gibbs' oscillations.

(d) (7 Points) Your spectrum analyzer does not output the DTFT, but only the DFT.

- (4 Points) Provide a sketch of the 42-point DFT of the first 42 samples of the original signal $x[n]$, (that is, for $n = 0, 1, 2, \ldots, 41$).
- (3 Points) For $M = 56$, consider the signal $y[n]$. Suppose we zero-pad the signal to length 84 (that is, we append zeroes to the signal such as to obtain a resulting signal of length 84), and we look at the 84-point DFT of the resulting signal. Explain the difference with respect to the first bullet item. Which gives a "better" output, and why? Explain your observations. *Hint:* A very rough sketch may help.

$\frac{0}{4}$

$$X(k) = \sum_{n=0}^{41} x(n) W_{42}^{kn}$$

$$X(k) = \sum_{n=0}^{41} \left[\frac{1}{\pi} \left(\cos\left(\frac{3\pi}{7} n\right) + \cos\left(\frac{4\pi}{7} n\right) \right) \right] W_{42}^{kn} = ?$$

The 84-point DFT is superior because it will oversample the DTFT and provide more information about the signal.

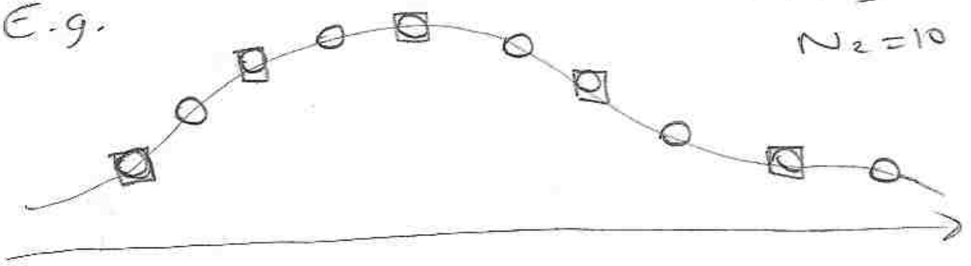
IN GENERAL, YES. BUT HERE, WE HAVE THE M=56 CASE...

$\frac{1}{3}$

E.g.

$N_1 = 5$ □

$N_2 = 10$ ○



The O's clearly provide more information about the signal.

Problem 4 *(Further problems with that digital TV set...)* 25 Points

When Tart Vimpson plugged in his new digital TV, he was dismayed to see ghosts on the screen. Some measurements revealed that this problem could be modeled by his digital video signal being filtered by an LTI system with impulse response

$$h(n) \;=\; \delta(n) - 0.1\delta(n-64) \tag{12}$$

To correct the problem, he wishes to process his signal by an inverse filter with impulse response $g(n)$, so that the effect of $h(n)$ is canceled. To determine $g(n)$ he computes the N-point DFT $\{H(k)\}_{k=0}^{N-1}$, with $N = 128$, of the sequence $h(n)$ and then defines $g(n)$ as the inverse DFT of $G(k) = 1/H(k)$, for $k = 0, 1, \ldots, N-1$.

(a) (11 Points) Determine $g[n]$, the 128-point inverse of $h[n]$.

$H(k) = 1 - 0.1 W_N^{64k} \Rightarrow 1 - 0.1 e^{-j\frac{2\pi}{128}\cdot 64k} = 1 - 0.1 e^{-j\pi k}$

$G(k) = \frac{1}{1 - \frac{1}{10}e^{-j\pi k}} = \frac{1 + \frac{1}{10}e^{-j\pi k}}{1 - \frac{1}{100}e^{-j2\pi k}} = \frac{1 + \frac{1}{10}e^{-j\pi k}}{99/100} = \frac{100}{99} + \frac{10}{99}e^{-j\pi k}$

$\boxed{g(n) = \frac{100}{99}\delta(n) + \frac{10}{99}\delta(n-64)}$ ✓

11/11

(b) (7 Points Detemine the linear convolution, $g(n) * h(n)$. Is the system with $g(n)$ the inverse of the one with unit pulse response $h(n)$? Why or why not?

$(g * h)(n) = g(n) - \frac{1}{10} g(n-64) = \frac{100}{99}\delta(n) - \frac{1}{99}\delta(n-128)$

$\boxed{(g * h)(n) = \frac{100}{99}\delta(n) - \frac{1}{99}\delta(n-128)}$

$g(n)$ is not the inverse of $h(n)$ because $(g * h)(n) \neq \delta(n)$, which is required for inverse responses. ✓

7/7

(c) (7 Points) Find the transfer function of an exact inverse. Find the causal sequence that corresponds to this transfer function.

$H(z) = 1 - 0.1z^{-64} \Rightarrow G(z) = \frac{1}{1 - 0.1z^{-64}} = \frac{z^{64}}{z^{64} - 0.1}$

$\boxed{H(z) = \frac{z^{64}}{z^{64} - 0.1}}$ ✓

7/7

$h(n+64) - 0.1h(n) = \delta(n+64) \rightarrow h(n) = \delta(n) + 0.1h(n-64)$

$h(n) = \delta(n) + 0.1[\delta(n-64)] \ldots$

$\boxed{h(n) = \sum_{k=0}^{\infty} 0.1^k \delta(n-64k)}$ ✓