You have two hours to complete this exam.

There are 100 points for this exam. Points for the individual problems and subproblems are marked in the problem statement.

The exam is closed-book and closed-notes; calculators, computing and communication devices are not permitted.

However, one handwritten and not photocopied single-sided sheet of notes is allowed.

No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

If we can’t read it, we can’t grade it.

We can only give partial credit if you write out your derivations and reasoning in detail.

You may use the back of the pages of the exam if you need more space.

*** GOOD LUCK! ***
Problem 1 *(Short questions.)*

For each of the following statements, decide whether they are true or false. If you believe a statement is true, give a proof. If you believe a statement is false, give a counterexample.

(a) *(5 Pts)* Consider a communication channel as in Figure 1 with input $X$, output $Y$, and conditional probability mass function $p(y|x)$. Then it is true that $H(Y) \geq H(X)$.

\[ X \quad p(y|x) \quad Y \]

Figure 1: A communication channel.

(b) *(5 Pts)* $D$-ary Huffman codes always satisfy the Kraft inequality with equality ($D \geq 2$).
(c) (5 Pts) Let \( X \) and \( Y \) be two zero-mean jointly Gaussian random variables. Let \( Z = X - \frac{E[XY]}{E[Y^2]} Y \). Then, \( Z \) and \( Y \) are independent.

\( (d) \ (5 \text{ Pts}) \) Suppose \( \{u_i(t)\}_{i=1}^N \) and \( \{v_j(t)\}_{j=1}^N \) are two orthonormal bases for the same space \( S \). Then if

\[
 s(t) = \sum_{i=1}^N \alpha_i u_i(t) = \sum_{i=1}^N \beta_j v_j(t) \tag{1}
\]

we have \( \sum_{i=1}^N |\alpha_i|^2 = \sum_{j=1}^N |\beta_j|^2 \).
Problem 2 (Source Coding with Side Information.)

A discrete memoryless source produces, in each time slot, a vector \((X, Y)\) with joint distribution \(p_{X,Y}(x,y)\) as follows:

\[
\begin{array}{c|cc}
  p(x,y) & y = \alpha & y = \beta \\
  x = a & 1/6 & 1/6 \\
  x = b & 1/12 & 1/6 \\
  x = c & 1/24 & 1/6 \\
  x = d & 1/24 & 1/6 \\
\end{array}
\]

As illustrated in Figure 2, \(Y\) is given both to the encoder and to the decoder, but \(X\) is only observed by the encoder. Your task is to devise an algorithm to be used by the encoder. The output of the encoder is a sequence of bits (i.e., of zeros and ones) such as to enable the decoder to get to know \(X\).

Remark. First, read all the subproblems (a)-(d). Then, start solving them.

(a) (10 Pts) Determine \(H(X|Y = \alpha), H(Y|X = b), \) and \(H(X|Y).\)
(b) (8 Pts) Develop an efficient prefix-free source encoding/decoding algorithm that maps each 
(x, y) pair independently into a uniquely decodable short sequence of bits. Carefully describe 
each step taken by the encoder and the decoder.

<table>
<thead>
<tr>
<th>p(x, y)</th>
<th>y = α</th>
<th>y = β</th>
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<tbody>
<tr>
<td>x = a</td>
<td>1/6</td>
<td>1/6</td>
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<tr>
<td>x = b</td>
<td>1/12</td>
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<td>x = c</td>
<td>1/24</td>
<td>1/6</td>
</tr>
<tr>
<td>x = d</td>
<td>1/24</td>
<td>1/6</td>
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(c) (5 Pts) Suppose your encoder from Part (b) outputs the bit string: 001010010101001010011101111111101...
At the decoder, you observe the following sequence of Y values: β, β, α, β, α, α, β. Decode the 
first 7 X symbols, using the coding scheme you devised in Part (b). Note: You may not need 
to decode all the bits in the string.

(d) (7 Pts) Determine the average number of bits, \( L \), that your encoder from Part (b) produces 
for each source output symbol. Show your derivation. It is not sufficient to merely give a number.
Problem 3 (Quantization.)

Consider a memoryless source whose outputs $X$ are uniformly distributed over the interval $[0, 2]$. In this problem, you will design quantizers for this source. The overall system looks like in Figure 3.

(a) (5 Pts) Find the best 1-bit scalar quantizer (quantization cell boundary and reconstruction points), determine the resulting mean-squared error distortion, and draw the corresponding point into the rate vs. distortion plot in Figure 4.

(b) (5 Pts) Find the best 2-bit scalar quantizer (quantization cell boundaries and reconstruction points), determine the resulting mean-squared error distortion, and draw the corresponding point into the rate vs. distortion plot in Figure 4.
(c) (5 Pts) Find the best $R$-bit scalar quantizer (quantization cell boundaries and reconstruction points), where $R$ is a positive integer, and determine the resulting mean-squared error distortion. Sketch the resulting behavior into the rate vs. distortion plot in Figure 4. *Hint:* Do all your calculations for a scalar quantizer with $M$ cells. What is the relationship between $R$ and $M$? At the very end, use this relationship to plot the distortion vs. $R$. 

Figure 4: Draw the solutions to Parts (a)-(e) here. Select an appropriate scale.
(d) (5 Pts) Consider the memoryless source whose outputs $Y$ are distributed according to the probability density function $f_Y(y)$ illustrated in Figure 5. We want to use a two-cell quantizer that assigns quantization indices as follows:

$$Q(y) = \begin{cases} 
0, & \text{if } 0 \leq y \leq \alpha \\
1, & \text{if } \alpha < y \leq 3
\end{cases}$$ \hspace{1cm} (2)

where $\alpha$ is chosen such as to minimize the mean-squared error of the source reconstruction based on the quantization index. Will the best $\alpha$ lie between 0 and 2, or between 2 and 3? Justify your answer with a mathematical argument. Hint: Start with the initial choice $\alpha_0 = 2$ and continue from there.

(e) (5 Pts) (Solve this problem after solving all other problems.) For the same setup as in Part (d), determine explicitly the optimal value of $\alpha$. Remark: Start by giving an outline of your derivation of the optimal $\alpha$, and carry out the calculations as far as you can.
Problem 4 (Signal Space.)

The following set of four waveforms is to be used for transmission across a channel that adds white Gaussian noise of power spectral density $N_0/2$ (that’s exactly the noise process that we considered in class). Assume that $N_0 = 1$ and that $b = \sqrt{6/T}$.

(a) (6 Pts) Draw an orthonormal basis for this set of waveforms. How many dimensions are there? Briefly show that your basis functions are orthonormal. **Hint:** Gram-Schmidt may not be the simplest solution.

(b) (6 Pts) Sketch the signal space characterization of this set of waveforms, including the signal points $s_1, s_2, s_3, s_4$.

(c) (3 Pts) Determine the average energy of the waveforms.
(d) (10 Pts) Now suppose that the signals of Figure 6 are used for transmission across a channel that adds white Gaussian noise of power spectral density $N_0/2$ (that’s exactly the noise process that we considered in class). Denote the received signal vector by $r$.

For the special case when $\text{Prob}(s_1) = 1/3$, $\text{Prob}(s_2) = 2/3$, and $\text{Prob}(s_3) = \text{Prob}(s_4) = 0$, determine the MAP decoding rule, using formulas. Then, sketch the MAP decoding rule into the figure you have drawn in Part (b). Use $b = \sqrt{6/T}$ and $N_0 = 1$ as before. You may make the approximation $\ln 2 \approx 0.7$. 
