# First Midterm Exam

| Last name | First name | SID |
|---|---|---|
| | | |

- You have two hours to complete this exam.
- There are 100 points for this exam. Points for the individual problems and subproblems are marked in the problem statement.
- The exam is closed-book and closed-notes; calculators, computing and communication devices are *not* permitted.
- However, one handwritten and *not photocopied* single-sided sheet of notes is allowed.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- If we can't read it, we can't grade it.
- We can only give partial credit if you write out your derivations and reasoning in detail.
- You may use the back of the pages of the exam if you need more space.

*** Good Luck! ***

| Problem | Points earned | out of |
|---|---|---|
| Problem 1 | | 20 |
| Problem 2 | | 30 |
| Problem 3 | | 25 |
| Problem 4 | | 25 |
| Total | | 100 |

**Problem 1** *(Short questions.)* 20 Points

For each of the following statements, decide whether they are true of false. If you believe a statement is true, give a proof. If you believe a statement is false, give a counterexample.

*(a) (5 Pts)* Consider a communication channel as in Figure 1 with input $X$, output $Y$, and conditional probability mass function $p(y|x)$. Then it is true that $H(Y) \geq H(X)$.

$X$ $p(y|x)$ $Y$

Figure 1: A communication channel.

*(b) (5 Pts)* $D$-ary Huffman codes always satisfy the Kraft inequality with equality ($D \geq 2$).

*(c) (5 Pts)* Let $X$ and $Y$ be two zero-mean jointly Gaussian random variables. Let $Z = X - \frac{E[XY]}{E[Y^2]}Y$. Then, $Z$ and $Y$ are independent.

*(d) (5 Pts)* Suppose $\{u_i(t)\}_{i=1}^{N}$ and $\{v_j(t)\}_{j=1}^{N}$ are two orthonormal bases for the same space $S$. Then if

$$s(t) = \sum_{i=1}^{N} \alpha_i u_i(t) = \sum_{i=1}^{N} \beta_j v_j(t) \qquad (1)$$

we have $\sum_{i=1}^{N} |\alpha_i|^2 = \sum_{j=1}^{N} |\beta_j|^2$.

**Problem 2** *(Source Coding with Side Information.)* 30 Points

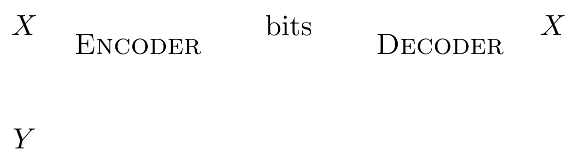


Figure 2: Source coding with side information.

A discrete memoryless source produces, in each time slot, a vector $(X, Y)$ with joint distribution $p_{X,Y}(x, y)$ as follows:

| $p(x,y)$ | $y=\alpha$ | $y=\beta$ |
|---|---|---|
| $x=a$ | $1/6$ | $1/6$ |
| $x=b$ | $1/12$ | $1/6$ |
| $x=c$ | $1/24$ | $1/6$ |
| $x=d$ | $1/24$ | $1/6$ |

As illustrated in Figure 2, $Y$ is given both to the encoder and to the decoder, but $X$ is only observed by the encoder. Your task is to devise an algorithm to be used by the encoder. The output of the encoder is a sequence of bits (i.e., of zeros and ones) such as to enable the decoder to get to know $X$.

*Remark. First,* read all the subproblems *(a)-(d). Then,* start solving them.

*(a) (10 Pts)* Determine $H(X|Y=\alpha), H(Y|X=b),$ and $H(X|Y)$.

*(b) (8 Pts)* Develop an efficient prefix-free source encoding/decoding algorithm that maps each $(x, y)$ pair independently into a uniquely decodable short sequence of bits. Carefully describe each step taken by the encoder and the decoder.

| $p(x,y)$ | $y=\alpha$ | $y=\beta$ |
|---|---|---|
| $x=a$ | 1/6 | 1/6 |
| $x=b$ | 1/12 | 1/6 |
| $x=c$ | 1/24 | 1/6 |
| $x=d$ | 1/24 | 1/6 |

*(c) (5 Pts)* Suppose your encoder from Part *(b)* outputs the bit string: 0010100101001010001011101...
At the decoder, you observe the following sequence of $Y$ values: $\beta, \beta, \alpha, \beta, \alpha, \alpha, \beta$. Decode the first 7 $X$ symbols, using the coding scheme you devised in Part *(b)*. *Note:* You may not need to decode all the bits in the string.

*(d) (7 Pts)* Determine the average number of bits, $\overline{L}$, that your encoder from Part *(b)* produces for each source output symbol. Show your derivation. It is not sufficient to merely give a number.

**Problem 3** *(Quantization.)* 25 Points

$X$ QUANTIZER quantization index (bits) RECONSTRUCTION $\hat{X}$

Figure 3: Quantization.

Consider a memoryless source whose outputs $X$ are uniformly distributed over the interval $[0, 2]$. In this problem, you will design quantizers for this source. The overall system looks like in Figure 3.

*(a) (5 Pts)* Find the best 1-bit scalar quantizer (quantization cell boundary and reconstruction points), determine the resulting mean-squared error distortion, and draw the corresponding point into the rate vs. distortion plot in Figure 4.

*(b) (5 Pts)* Find the best 2-bit scalar quantizer (quantization cell boundaries and reconstruction points), determine the resulting mean-squared error distortion, and draw the corresponding point into the rate vs. distortion plot in Figure 4.

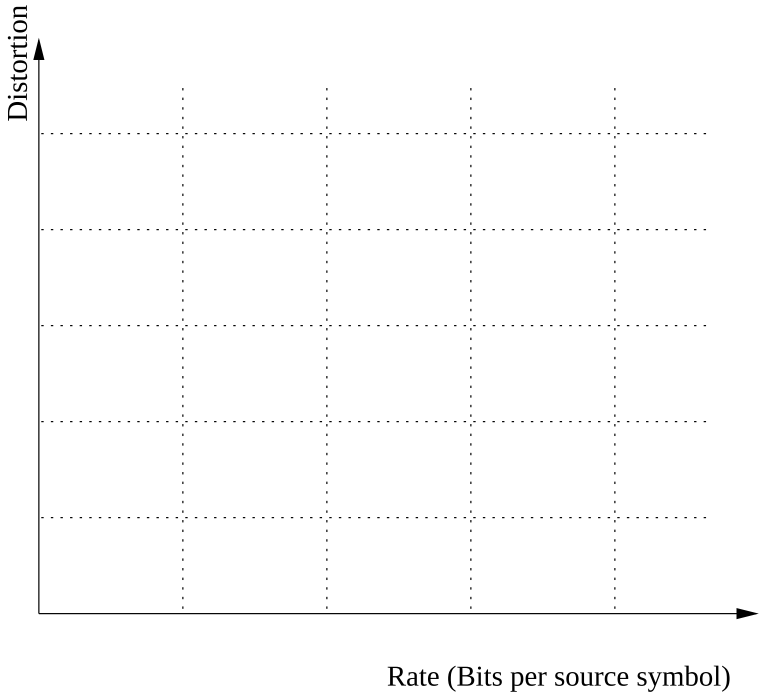


Figure 4: Draw the solutions to Parts *(a)*-*(c)* here. Select an appropriate scale.

*(c) (5 Pts)* Find the best $R$-bit scalar quantizer (quantization cell boundaries and reconstruction points), where $R$ is a positive integer, and determine the resulting mean-squared error distortion. Sketch the resulting behavior into the rate vs. distortion plot in Figure 4. *Hint:* Do all your calculations for a scalar quantizer with $M$ cells. What is the relationship between $R$ and $M$? At the very end, use this relationship to plot the distortion vs. R.

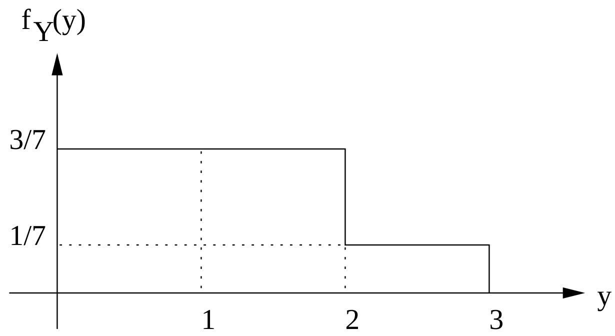


Figure 5: The source distribution for Parts *(d)* and *(e)*.

*(d) (5 Pts)* Consider the memoryless source whose outputs $Y$ are distributed according to the probability density function $f_Y(y)$ illustrated in Figure 5. We want to use a two-cell quantizer that assigns quantization indices as follows:

$$Q(y) \quad = \quad \begin{cases} 0, & \text{if } 0 \le y \le \alpha \\ 1, & \text{if } \alpha < y \le 3, \end{cases} \tag{2}$$

where $\alpha$ is chosen such as to minimize the mean-squared error of the source reconstruction based on the quantization index. Will the best $\alpha$ lie between $0$ and $2$, or between $2$ and $3$? Justify your answer with a *mathematical* argument. *Hint:* Start with the initial choice $\alpha_0 = 2$ and continue from there.

*(e) (5 Pts) (Solve this problem after solving all other problems.)* For the same setup as in Part *(d)*, determine explicitly the optimal value of $\alpha$. *Remark:* Start by giving an outline of your derivation of the optimal $\alpha$, and carry out the calculations as far as you can.

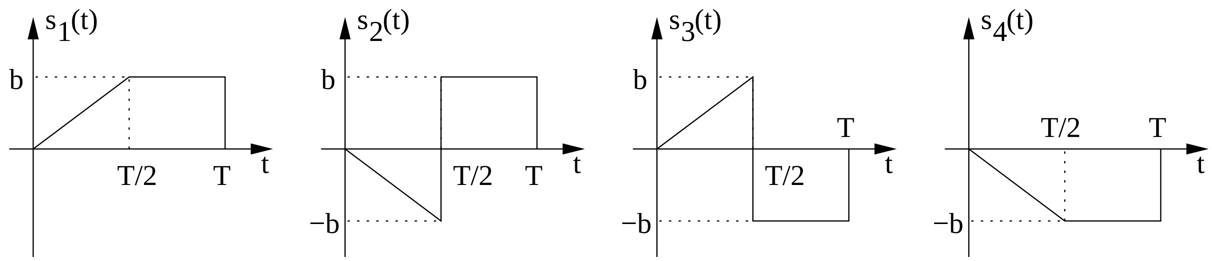


Figure 6: The waveforms for Problem 4. Here, $b = \sqrt{6/T}$.

**Problem 4** *(Signal Space.)* 25 Points

The following set of four waveforms is to be used for transmission across a channel that adds white Gaussian noise of power spectral density $N_0/2$ (that's exactly the noise process that we considered in class). Assume that $N_0 = 1$ and that $b = \sqrt{6/T}$.

*(a) (6 Pts)* Draw an orthonormal basis for this set of waveforms. How many dimensions are there? Briefly show that your basis functions are orthonormal. *Hint:* Gram-Schmidt may not be the simplest solution.

*(b) (6 Pts)* Sketch the signal space characterization of this set of waveforms, including the signal points $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$.

*(c) (3 Pts)* Determine the average energy of the waveforms.

*(d) (10 Pts)* Now suppose that the signals of Figure 6 are used for transmission across a channel that adds white Gaussian noise of power spectral density $N_0/2$ (that's exactly the noise process that we considered in class). Denote the received signal vector by $\mathbf{r}$.

For the special case when $Prob(\mathbf{s}_1) = 1/3, Prob(\mathbf{s}_2) = 2/3$, and $Prob(\mathbf{s}_3) = Prob(\mathbf{s}_4) = 0$, determine the MAP decoding rule, using formulas. Then, sketch the MAP decoding rule into the figure you have drawn in Part *(b)*. Use $b = \sqrt{6/T}$ and $N_0 = 1$ as before. You may make the approximation $\ln 2 \approx 0.7$.