Final Exam

The examination is for 180 minutes. The maximum score is 90 points. Your answers should be unambiguous.

1. (*4 + 4 + 6 points*)

   State whether the following statements are true or false, and give a reason for your answer. A correct answer with an incorrect reason will get only 1 point.

   (a) Given real valued random variables X and Y, let $Z := X + Y$. The characteristic function of Z is the product of the characteristic function of X with the characteristic function of Y.

   (b) Let $m(t)$ be a real valued baseband signal with bandwidth $W << f_c$, and let $x(t) := Am(t)\cos(2\pi f_c t)$, where $A > 0$ is a fixed constant. Thus $x(t)$ is a DSB-SC signal corresponding to the modulating signal $m(t)$ and carrier frequency f_c. Let $\hat{x}(t)$ denote the Hilbert transform of $x(t)$. Then $\hat{x}(t) = A\hat{m}(t)\sin(2\pi f_c t)$, where $\hat{m}(t)$ denotes the Hilbert transform of $m(t)$.

   (c) Consider communicating using baseband binary PAM over a non-ideal channel with real valued impulse response $c(t)$. If we are willing to communicate at a slow enough rate we can find real valued transmit and receive filters so that communication is possible with zero ISI.

2. (*5 points*)

   A linear time invariant system has the property that when the input to the system is the pure tone $\cos(2\pi f t)$, the output is $\sin(2\pi f t)$. What is the transfer function of this system ? Explain your answer.

3. (*8 points*)

   $(X_n, n \in \mathbf{Z})$ is a real valued stationary stochastic process. $(A_n, n \in Z)$ is a WSS stochastic process, which is independent of $(X_n, n \in \mathbf{Z})$. Further, we have $A_n \geq 1$ for all $n \in \mathbf{Z}$.

   Let $Y_n := X_n + A_n X_{n-1}$.

   Is $(Y_n, n \in \mathbf{Z})$ a WSS process ? Either argue why this is the case or argue why this is false.

4. (*6 + 6 points*)

   Let X be a random variable taking values in the finite set $\mathcal{X}$. Let L denote the mean length of a Huffman code for X.

Let $Z := (X_1, X_2)$, where X_1 and X_2 are independent random variables, each of which has the same distribution as X. Thus, Z takes values in the finite set

$$\mathcal{X} \times \mathcal{X} := \{(x_1, x_2) \ : \ x_1, x_2 \in \mathcal{X}\} \ .$$

Let $\tilde{L}$ denote the mean length of a Huffman code for Z.

(a) Argue that

$$\tilde{L} \leq 2L \ .$$

(b) Give an example where $\tilde{L} < 2L$ (strict inequality !).

5. (*7 points*)

Consider the following four energy type signals :

$$\begin{aligned}
s_1(t) &= \Lambda(\frac{2t}{T} - 1) \ , \\
s_2(t) &= \begin{cases} \frac{4}{T}t & \text{if } 0 \leq t \leq \frac{T}{4} \\ 1 & \text{if } \frac{T}{4} \leq t < \frac{T}{2} \\ 0 & \text{elsewhere} \end{cases} \ , \\
s_3(t) &= \begin{cases} 1 & \text{if } \frac{T}{2} < t \leq \frac{3T}{4} \\ \frac{4}{T}(T-t) & \text{if } \frac{3T}{4} \leq t \leq T \\ 0 & \text{elsewhere} \end{cases} \ , \\
s_4(t) &= \Lambda(\frac{4t}{T} - 2) \ , .
\end{aligned}$$

Find a smallest possible set of orthogonal unit energy signals such that each of these four signals can be written as a linear combination of the signals in this set.

You need not use the Gram Schmidt orthogonalization procedure unless you want to. However, you should explain why the set of signals you propose is the smallest such set.

6. (*4 + 6 + 6 points*)

Consider the signaling constellation shown in Figure 1.

It is comprised of 16 points (represented by the thick dots in the figure). The small dot in the middle is NOT a constellation point : it is there just to indicate the location of the origin $(0,0)$ in the coordinate system in signal space. The grid shown can be taken to have side length A. If the constellation point $[x_1, x_2]^T$ is sent, the received vector at the output of the matched filter bank may be written as

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \ ,$$

where n_1 and n_1 are independent zero mean Gaussian random variables, each having variance $\frac{N_0}{2}$.

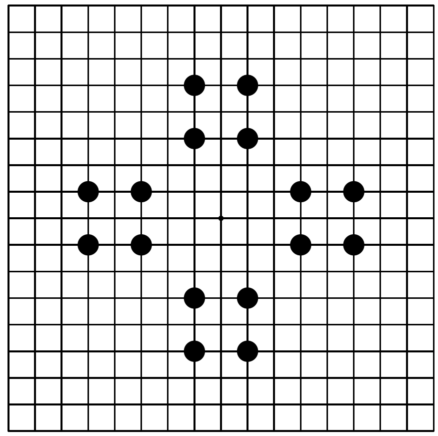

Figure 1: Signaling constellation

(a) What is the probability of error of maximum likelihood symbol by symbol decoding, conditioned on the transmitter sending the signal corresponding to the constellation point $(5A, A)$ in a symbol interval ?

(b) What is the probability of error of maximum likelihood symbol by symbol decoding, conditioned on the transmitter sending the signal corresponding to the constellation point $(-3A, -A)$ in a symbol interval ? It is enough to leave the answer in the form of an integral.

(c) A Stanford student claims that in the infinite SNR limit the ratio of the quantities determined in the two previous parts approaches 1. Is this true or false ? Explain your answer.

7. (*10 points*)

Consider QPSK signaling with bit energy E_b and assume that the transmitted signals over a symbol interval are

$$\pm\sqrt{\frac{2E_b}{T}}\cos(2\pi f_c t) \pm \sqrt{\frac{2E_b}{T}}\sin(2\pi f_c t)\ ,$$

where T denotes the length of a symbol interval. Also, assume that f_c is an integer multiple of $\frac{1}{T}$.

The transmitted signal is received in additive white Gaussian noise of power spectral density $\frac{N_0}{2}$. The receiver operates symbol by symbol using matched filtering and the MLE optimal decision regions based on the sampled outputs of the matched filter bank.

Suppose that the local oscillator at the receiver has a phase error. Namely, instead of correlating the received signal over a symbol interval with $\sqrt{\frac{2}{T}}\cos(2\pi f_c t)$

and $\sqrt{\frac{2}{T}}\sin(2\pi f_c t)$ respectively, the correlation is done with $\sqrt{\frac{2}{T}}\cos(2\pi f_c t+\theta)$ and $\sqrt{\frac{2}{T}}\sin(2\pi f_c t+\theta)$ respectively, where θ is some fixed phase error. It is not known at the receiver that there is a phase error, so the decision regions of the receiver are the same as they would be if there were no phase error. Assume that the phase error satisfies $0 < \theta < \frac{\pi}{4}$.

What is the average symbol probability at the receiver ?

8. (*4 + 4 points*)

   Let Z be Rayleigh distributed. Let Θ be independent of Z and uniformly distributed on $[0, 2\pi)$.

   (a) What is the distribution of $Z\cos\Theta$?

   (b) Argue that $Z\cos\Theta$ and $Z\sin\Theta$ are independent and identically distributed random variables.

   Hint : If you understand how Rayleigh distributions arise, it will not be necessary to do any calculations to satisfactorily answer this problem.

9. (*10 points*)

   Consider the rate $\frac{2}{3}$ convolutional code with encoder as shown in Figure 2. For convenience, the equations relating the outputs to the inputs are given below :

$$
\begin{aligned}
y_k^{(1)} &= x_k^{(1)} + x_{k-1}^{(1)} + x_{k-1}^{(2)} \ , \\
y_k^{(2)} &= x_{k-1}^{(1)} + x_{k-1}^{(2)} \ , \\
y_k^{(3)} &= x_k^{(2)} + x_{k-1}^{(1)} + x_{k-1}^{(2)} \ .
\end{aligned}
$$

   Draw the state transition diagram for the encoder. Your diagram should have four states and sixteen directed edges. Show the labels on at least three distinct edges in your diagram. Your label should indicate the input bit pair and output bit triplet corresponding to the edge.

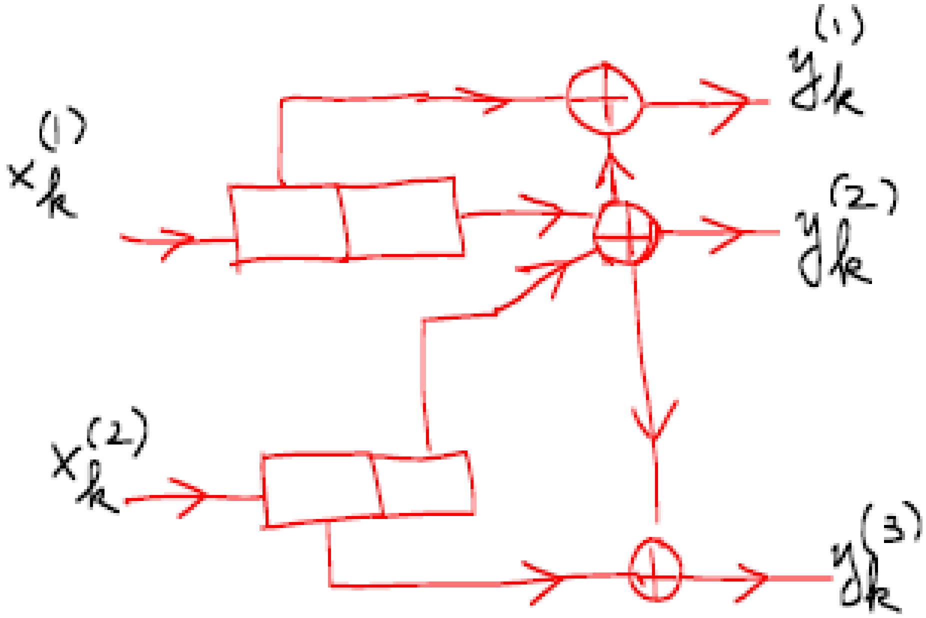

FIG. 2