Problem #1

1. 
\[ E_0(f) = |H(f)|^2 * S_{ww}(f) = N_0 A^2 / 2 * \pi f / 2w \]
\[ v(t) = u(t) \sin(2 \pi f_0 t) \]
\[ R_u(t + \tau, t) = E[v(t + \tau) \cdot v(t)] \]
\[ = E[u(t + \tau) \cdot u(t) \sin(2 \pi f_0 (t + \tau)) \sin(2 \pi f_0 t)] \]
\[ = 0.5 E[u(t + \tau) \cdot u(t) \cos(2 \pi f_0 \tau) - \cos(4 \pi f_0 t + 2 \pi f_0 \tau)] \]
\[ = 0.5 R_u(\tau) \cos(2 \pi f_0 \tau) - 0.5 R_u(\tau) \cos(4 \pi f_0 t + 2 \pi f_0 \tau) \]
\[ \frac{1}{T_0} \int_{t'}^{t+T_0} R_u(t + \tau, t) d\tau = R_u(\tau) \cos(2\pi * f_0 * \tau) \]
\[ S_v(f) = S_u(f) * [0.5 \delta(f - f_0) + 0.5 \delta(f + f_0)] \]
\[ = 0.5 S_u(f - f_0) + 0.5 S_u(f + f_0) \]
\[ = N_0 A^2 / 8 * \pi(f - f_0) / 2w + N_0 A^2 / 8 * \pi(f + f_0) / 2w \]
\[ S_w(f) = |H(f)|^2 * S_v(f) \]
\[ = N_0 A^2 / 8 * \pi(f / 2w) * \pi(f - f_0) / 2w + N_0 A^2 / 8 * \pi(f / 2w) * \pi(f + f_0) / 2w \]

Problem #2

2. 
\[ V(f) = M(f) * [1/2j * \delta(f - f_1) - 1/2j * \delta(f + f_1)] \]
\[ = 1/2j * M(f - f_1) - 1/2j * M(f + f_1) \]

Since \( f_1 \gg W \) the Fourier Transform of the corresponding analytic signal is 
\[ Z(f) = 1/j * M(f - f_1) \]
\[ z(t) = 1/j * m(t) e^{j*2\pi f_1 t} \]
\[ u_d(t) = 1/j * m(t) e^{j*2\pi(f_1 - f_0)t} \]
\[ = m(t) \sin(2\pi(f_1 - f_0)t) - jm(t)\cos(2\pi(f_1 - f_0)t) \]

Hence 
\[ u_d(t) = m(t) \sin(2\pi(f_1 - f_0)t) \]
\[ u_s(t) = -m(t) \cos(2\pi(f_1 - f_0)t) \]

Note: It is irrelevant whether \( f_0 \gg W \) or that \( |f_1 - f_0| \) is small relative to \( f_1 \) and \( f_0 \)
Problem #3

3. 

\[ S(t) \text{ is even, so} \]

\[ s(t) = s_0/2 + \sum_{n=1}^{\infty} s_n \cos(2\pi n f_c t) \]

for some \( s_0, s_1, s_2 \ldots \)

\[ u(t) = m(t) s(t) = s_0/2 * m(t) + \sum_{n=1}^{\infty} s_n m(t) \cos(2\pi n f_c t) \]

\[ v(t) = s_1 m(t) \cos(2\pi f_c t) \text{ because } f_c >> W \]

\( n(t) \) is bandpass noise with flat power spectral density \( N_0/2 \) over the bands \( \pm f_c \pm W \)

\[ r(t) = s_1 m(t) \cos(2\pi f_c t) + n_c(t) \cos(2\pi f_c t) - n_s(t) \sin(2\pi f_c t) \]

\[ r(t) \cos(2\pi f_c t) \text{ after low pass filtering yields } 1/2\left[ s_1 m(t) + n_c(t) \right] \]

The message signal power at the output is

\[ P_0 = 1/4 * s_1^2 * P_M \]

The noise power at the output is

\[ P_n = 1/4 * P_{nc} = 1/4 * N_0/2 * 4W = N_0W/2 \]

\( (S/N)_0 = P_0/P_n = s_1^2 P_M/2WN_0 \)

Here \( s_1 = 2T_c \left[ \int_{-T_c/4}^{T_c/4} A_0 \cos(2\pi f_c t) dt \right. \]

\[ - \int_{-T_c/2}^{T_c/2} A_0 \cos(2\pi f_c t) dt \]

\[- \int_{-T_c/4}^{T_c/4} A_0 \cos(2\pi f_c t) dt \]

\[ = 4/\pi^2 A_0 \]

The desired power is

\[ P_R = \sum_{n=1}^{\infty} s_n^2 P_M \text{ because } f_c >> W \text{ and because } s_0 = 0 \]

By Parseval's relation, since \( s_0 = 0 \)

\[ \sum_{n=1}^{\infty} s_n^2 = A_0^2 \]

Thus

\[ P_R = A_0^2 * P_M \]

Hence \( (S/N)_0 = (16/\pi^2) * (A_0^2 * P_M/2WN_0) = 8/\pi^2 * P_R/WN_0 \)

\[ = 8/\pi^2 * (S/N)_b \]

Problem #4
4. Let $x_1 < x_2$ be the quantization levels chose. Let us write $u = (x_1 + x_2)/2$ and

\[ x_1 = u - \nu \]
\[ x_2 = u + \nu \]

Given $u$, the choice of $\nu$ is decide by

(A) $\int_{-\infty}^{\mu} x \phi(x) dx = (u - \nu) \int_{-\infty}^{\mu} \phi(x) dx$

(B) $\int_{-\infty}^{\mu} x \phi(x) dx = (u + \nu) \int_{-\infty}^{\mu} \phi(x) dx$

where $\rho(\nu)$ derives the Gaussian density

$$\frac{1}{(\sqrt{2\pi}\sigma)} e^{-\nu^2/2\sigma^2}$$

The mean square distribution is

\[ \int_{-\infty}^{\mu} (x - (u - \nu))^2 \phi(x) dx + \int_{-\infty}^{\mu} (x - (u + \nu))^2 \phi(x) dx \]

\[ = \int_{-\infty}^{\mu} x^2 \phi(x) dx - (u - \nu)^2 \int_{-\infty}^{\mu} \phi(x) dx + \int_{-\infty}^{\mu} x^2 \phi(x) dx - (u + \nu)^2 \int_{-\infty}^{\mu} \phi(x) dx \]

where we used (A) and (B)

using (A) and (B) again, this can be written as

\[ = \sigma^2 - ((\int_{-\infty}^{\mu} x \phi(x) dx)^2 + (\int_{-\infty}^{\mu} \phi(x) dx)^2) \]

so we want to choose $u$ to maximize

\[ (\int_{-\infty}^{\mu} x \phi(x) dx)^2 + (\int_{-\infty}^{\mu} \phi(x) dx)^2 \]

clearly the best choice is $u = 0$

Then $\nu$ would be chosen so that $u - \nu = x_1$ is the centroid of the left half
and $u + \nu = x_2$ is the centroid of the right half of the density

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