Problem 1 (30 points)

[8 pts.] 1a) Argue that for any binary code satisfying the prefix-free condition, the codeword lengths \( \{l_i\} \) must satisfy the Kraft’s inequality:

\[
\sum_i 2^{-l_i} \leq 1.
\]

[6 pts.] b) Is it true that for any source, the codeword lengths for the binary Huffman code must satisfy Kraft’s inequality with equality? Explain.

[6 pts.] c) Suppose now the coded symbols are from a general alphabet of size \( D \). The Kraft inequality becomes:

\[
\sum_i D^{-l_i} \leq 1.
\]

Is it true that the Huffman code must satisfy Kraft’s inequality with equality? Explain.

[10 pts.] d) Consider a source for which the letter probabilities are of the form \( 2^{-k} \), where \( k \) is an integer. Construct the Huffman code and give the corresponding codeword lengths. Justify that the code is optimal.
Problem 2 (30 points)

Let \( \{X(t)\} \) be a zero-mean WSS Gaussian process with autocorrelation function \( R_x(\tau) = e^{-|\tau|} \).

[6 pts.] a) Find its power spectral density.

[8 pts.] b) Suppose we sample this process every \( T \) seconds. Is the resulting discrete-time process Gaussian? WSS? If so, compute its autocorrelation function.

[8 pts.] c) Let \( \{Y_n\} \) be the sampled process. We perform DPCM quantization by LLSE prediction of \( Y_n \) from \( Y_{n-1} \). Find the distribution of the residual error \( Y_n - \hat{Y}_n \).

[8 pts.] d) The residual error is quantized by a single bit quantizer to values \( \pm \Delta \). Find the optimal choice of \( \Delta \) as a function of \( T \). What happens when \( T \to 0 \)?
Problem 3 (20 points)

Here is one way to simulate white noise. Let \{W_n\} be iid. rv’s with \( P(W_n = 1) = P(W_n = -1) = \frac{1}{2} \). For each \( K \), define the continuous process \( \{W^{(k)}(t)\} \) for \( t \geq 0 \) as follows:

\[
W^{(k)}(t) = W_n \sqrt{K} \quad \text{for} \quad \frac{n}{K} \leq t \leq \frac{n + 1}{K}, \quad n = 0, 1, 2, \ldots
\]

For large \( K \), this can be used to approximate \( \{W(t)\} \).

[4 pts.] a) Sketch a typical sample path of \( \{W^{(k)}(t)\} \).

[10 pts.] b) Compute \( \text{Var}[W^{(k)}(t)] \) and \( \text{Var}\left[\int_0^1 W^{(k)}(t)\,dt\right] \). Based on this calculation, explain why while

\[
\text{Var}[W(t)] = \infty, \quad \text{Var}\left[\int_0^1 W(t)\,dt\right] \text{ is finite.}
\]

[6 pts.] c) A student does not like the fact that \( \text{Var}[W(t)] = \infty \). He decides to use instead the approximation

\[
Y^{(k)}(t) = W_n \quad \text{for} \quad \frac{n}{K} \leq t \leq \frac{n + 1}{K}, \quad n = 0, 1, 2
\]

What is wrong with this noise model for \( K \) large?
Problem 4 (20 points)

A data stream is partitioned into blocks of two bits. A block is modulated onto a signal waveform, say on $[0, 1]$. Consider two modulation schemes:

1) Scheme A: The two bits are modulated into a 4-level PAM, with the 4 levels equally spaced and symmetric about 0.

2) Scheme B: The signal waveform is composed by separately modulating each bit into a 2-level PAM. The waveform for the first bit is on $[0, \frac{1}{2}]$, and the one for the second bit on $[\frac{1}{2}, 1]$. The levels are symmetric about 0.

[6 pts.] a) Sketch the possible signal waveforms for both schemes.

[14 pts.] b) Find an orthonormal basis for each scheme (on $[0, 1]$). What are the dimensions of the signal space? Sketch a geometric representation of the signal constellation.

[6 pts.] c) BONUS: An important measure of a modulation scheme is the minimum distance between the constellation points. For a given minimum distance $d$, find the average energies required in both schemes. Which scheme is better in this respect? (You can assume that each of the four possible messages is equally likely.)