Problem 1 (10 points)

Sketch the following time functions, labelling maximum amplitude.

[4 pts.] a) \( \cos[2\pi t + 2\pi u(t)] \)

\[ t < 0: \cos(2\pi t) \quad t > 0: \cos(2\pi t) \]

[4 pts.] b) \( \frac{1}{2}(1 + u(t))\cos(2\pi t) \)

\[ t < 0: \frac{1}{2}\cos(2\pi t) \quad t > 0: \cos(2\pi t) \]

[2 pts.] c) What is the message \( m(t) \) in the signals above? What is the modulation method (e.g., USB, etc.)?

i) \( \cos[2\pi t + 2\pi u(t)] : m(t) = \frac{u(t)}{u(t)} \quad \text{method} = \text{FM} \)

ii) \( \frac{1}{2}(1 + u(t))\cos(2\pi t) : m(t) = \frac{u(t)}{u(t)} \quad \text{method} = \text{AM-DSB-SC} \)

Sketch \( X_1(f), Y(f), Z(f), Z_2(f) \), labelling important amplitudes and frequencies.

\( X_1(f) \):

\( Y(f) \):

\( Z(f) \):

Find \( f_s \) such that \( A_f \) is one of the 20 MHz segments (half circle, triangle, pulse, or arches).

\[ f_s = \frac{6000 \text{ or } 640}{\text{MHz}} \]

\( Z_2(f) \):

\[ f_0 = 660 \]
Problem 3 (25 points)

A system is described by the following block diagram:

\[ w(t) = \sum_{n=-\infty}^{\infty} \delta(t - 2n) \sin(\pi t / 3) \]

\[ W(f) = \begin{cases} 1 & \text{if } f = n - \frac{2}{3} \frac{1}{\sqrt{2}} \end{cases} \]

\[ \mathbf{X}(f) \] is the input and \( \mathbf{X}'(f) \) is the output.

\[ w(t) = \frac{\sin(\pi t / 3)}{\pi t} \]

[8 pts.] a) Is the system with input \( x(t) \) and output \( x'(t) \) (circle correct answer):

- causal? YES NO NOT DECIDABLE
- linear? YES NO NOT DECIDABLE
- time-invariant? YES NO NOT DECIDABLE
- BIBO stable? YES NO NOT DECIDABLE

[17 pts.] b) For \( x(t) = \cos(2\pi t) \), sketch the spectra \( X_1(f) \) and \( X'(f) \), labelling important amplitudes and frequencies.

\[ X_1(f) = \sum_1 N \frac{B}{2} \sum_{-\infty}^{\infty} s / (4 - \frac{2}{3}) \]

Problem 4 (25 points)

The differential equation for a broom balancing system below is given by:

\[ \ddot{\theta}(t) = \theta(t) - f(t) \]

in terms of block diagrams, and with zero initial conditions:

\[ F(s) = \frac{\theta(s)}{\theta(s)} = \frac{1}{s^2 + \frac{1}{2}} \]

[5 pts.] a) Find \( H(s) \). \( H(s) = \frac{\theta(s)}{F(s)} = \frac{\sqrt{\frac{1}{s^2 - 1}}}{s^2} \]

[10 pts.] b) Suppose we measure \( \theta(t) \) and use a feedback of the form \( f(t) = \alpha \theta(t) \) for \( \theta(0-) = 0 \) and \( \theta(0-) = \frac{1}{2} \). Find \( \alpha \) such that \( \theta(t) = \frac{1}{2} e^{-10t} u(t) \). If no such \( \alpha \) exists, explain why.

\[ \frac{\theta(t)}{\theta(t)} - \alpha \theta(t) \]

Since \( \alpha \) is chosen, we get one pole at \( s = -10 \) unstable.

\[ \text{roots of denominator = poles} \]

\[ S = \pm \frac{1}{\sqrt{1 - \alpha}} \]

Since \( \alpha = 1 \), we get one pole at \( s = 1 \) unstable.

[10 pts.] c) Using a feedback of the form \( f(t) = 11 \theta(t) + \beta \theta(t) \) and the same values from (b), above, find \( \beta \) such that \( \theta(t) \rightarrow 0 \) as \( t \rightarrow \infty \). Justify your answer.

\[ s^2 \theta(t) - s \theta(0-) - \theta(t) = (s^2 + \beta s + \frac{1}{2}) \theta(t) \]

Find value \( \beta \) for poles in LHP.
Problem 5  (35 points)

You wish to determine the impulse response of an auditorium, so that you can remove the echoes from a signal recorded in that room.

From measurements made in the auditorium, you know that if you generate an impulse at \( t = 0 \) s in one location, you hear that impulse at \( t = \frac{1}{2} \) s at your location, and echoes at 1 s intervals with primary echo at \( t = \frac{3}{2} \) s.

The amplitude of each successive echo is 20% that of the previous echo. (The primary echo at 3/2 sec has an amplitude of 20% of the direct impulse at \( t = \frac{1}{2} \) sec.)

[10 pts.] a) What is \( h(t) \)?

\[
\sum_{n=0}^{\infty} (z)^n e^{(t-\frac{3}{2})n}
\]

[5 pts.] b) What is \( H(z) \)?

\[
\sum_{n=0}^{\infty} (z)^n e^{-\frac{s}{2}n}
\]

for \( \text{Re}(z) > 0 \)

\[
H(z) = \frac{e^{-\frac{s}{2}}}{1 - 0.2e^{-s}}
\]

[10 pts.] c) With \( H(z) \) as above,

\[
x(t) \xrightarrow{H(z)} y(t) \xrightarrow{G(z)} z(t)
\]

Design \( G(z) \) such that \( z(t) = x(t) \). What is \( g(t) \)?

\[
G(z) = \frac{1 - 0.2e^{-s}}{e^{-\frac{s}{2}}} = e^{\frac{s}{2}h} - 0.2e^{\frac{s}{2}}
\]

\[
g(t) = \delta(t + \frac{3}{2}) - 0.2 \delta(t - \frac{1}{2})
\]

[10 pts.] d) With \( H(z) \) as above,

\[
x(t) \xrightarrow{H(z)} y(t) \xrightarrow{G(z)} z[n]
\]

Design \( G(z) \) such that \( z[n] = x(t)|_{t=nT} \). The switch represents sampling \( y(t) \) such that \( y[n] = y(nT) \). Use impulse invariance techniques to find \( h[n] \) and \( H(z) \).

\[
G(z) = \frac{T}{H(z)} = 2^2 \left( 1 - 2^2 e^{-4k} \right)
\]

\[
h[n] = \sum_{k=0}^{\infty} (z)^k \delta(n - 2(4k + 1))
\]

\[
H(z) = \sum_{k=0}^{\infty} (z)^k 2^{-2} e^{-4k}
\]

\[
= 2^{-2} \frac{2^{-2}}{1 - 2^2 e^{-4}}
\]
Problem 7 (22 points)

Given the system and the input $x[n]$, find the output $y[n]$

$h[n]$ is the unit sample response of a linear shift invariant system.

\[ \frac{1}{1 - \frac{1}{2}z^{-1}} \cdot \frac{1}{1 - \frac{1}{4}z^{-1}} = \frac{8/3}{1 - 4/15z^{-1}} + \frac{-5/3}{1 - 4/15z^{-1}} \]

3 pts. a) If $x[n] = (4/5)^n u[n]$ and $h[n] = \left( \frac{1}{2} \right)^n u[n]$, find $y[n]$.

\[ y[n] = \frac{8}{15} \left( \frac{4}{5} \right)^n u[n] - \frac{5}{15} \left( \frac{4}{5} \right)^n u[n] \]

3 pts. b) If $x[n] = e^{j\pi/4} u[n]$ and $h[n] = \delta[n] + \delta[n - 2]$, find $y[n]$.

\[ y[n] = e^{j\pi/4} u[n] + e^{j\pi/4} u[n-2] \]

3 pts. c) If $x[n] = e^{j\pi/4} u[n]$ and $h[n] = H(z) = \frac{z - 3/2}{z - 2/3}$, find $y[n]$.

\[ y[n] = e^{j\pi/4} \left( \frac{e^{j\pi/4} - 2/3}{e^{j\pi/4} - 3/2} \right) \]

3 pts. d) If $x[n] = (4/5)^n u[n]$ and $h[n] = \delta[n] + \frac{1}{2} \delta[n-1] + \frac{1}{4} \delta[n-2]$, find $y[n]$.

\[ y[n] = \left( \frac{4}{5} \right)^n u[n] + \frac{1}{2} \left( \frac{4}{5} \right)^{n-1} u[n-1] + \frac{1}{4} \left( \frac{4}{5} \right)^{n-2} u[n-2] \]

3 pts. e) If $x[n] = \delta[n]$ and the DTFT of $h[n] = H(e^{j\omega T}) = 1 + e^{-j\omega T}$, find $y[n]$.

\[ y[n] = \delta[n] + \delta[n-4] \]

\[ H[\omega] = 1 + e^{-j\omega T} \]

\[ H[\omega] = 1 + \delta[\omega - 4] \]

4 pts. f) Is $H(z) = \frac{1}{1 - 2z^{-1}}$ the transfer function for a BIBO stable system? Why or why not?

\[ H(z) = \frac{z}{z - 2} \]

Pole at $z = 2$.

For causal system to be BIBO, $\Re \{z\} < 1$

3 pts. g) If $H(z) = \frac{a - bz^{-1}}{c - dz^{-1}}$, what is the corresponding linear difference equation with input $x[n]$ and output $y[n]$?

\[ y[n] - ay[n-1] = bx[n] + cy[n-1] \]
hint: assume \( \pi(t) = \begin{cases} 
0 & t < -1/2, t > 1/2 \\
1/2 & t = -1/2, t = 1/2 \\
1 & -1/2 < t < 1/2 
\end{cases} \)

\( x_8[n] \)

or equivalently:

\( X_8[k] \) is sketch \[ \boxed{ \text{K} } \]