Instructions:

- There are four questions on this midterm. Answer each question in the space provided, and clearly label the parts of your answer. You can use the additional blank pages at the end for scratch paper if necessary. **Do NOT write answers on the back of any sheet or in the additional blank pages. Any such writing will NOT be graded.**

- Each problem is worth 25 points, and you may solve the problems in any order.

- Show all work. If you are asked to prove something specific, you must give a derivation and not quote a fact from your notes sheet. Otherwise, you may freely use facts and properties derived in class; just be clear about what you are doing!

- None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.

- You may use one double-sided sheet of notes. **No calculators are allowed** (or needed).
Problem 1. (DT Processing of a CT Signal) A bandlimited continuous time signal is known to contain a 60-Hz component, which we want to remove by processing with the system shown below.

a) Assuming $T = 10^{-4}$ s, what is the highest frequency that the continuous time input signal can contain if aliasing is to be avoided?

b) Draw the frequency responses of the ideal anti-aliasing and ideal reconstruction filters assuming $T = 10^{-4}$ s. You may assume the ADC produces samples $x[n] = x(nT)$ for integer values of $n$, and the DAC generates the weighted impulse train $y(t) = \sum_{n=-\infty}^{\infty} y[n] \delta(t - nT)$.

c) The discrete-time system to be used has frequency response

$$H(e^{j\omega}) = \frac{(1 - e^{-j(\omega - \omega_0)})(1 - e^{-j(\omega + \omega_0)})}{(1 - 0.9e^{-j(\omega - \omega_0)})(1 - 0.9e^{-j(\omega + \omega_0)})}.$$ 

Compute the filter gain $|H(e^{j\omega})|^2$, and make a rough sketch of it.

(Hint: $\frac{(1.8 - 1.8 \cos(\omega - \omega_0))}{(1.81 - 1.8 \cos(\omega - \omega_0))} \approx 1$ unless $\omega$ is very close to $\omega_0$...)

d) What value should be chosen for $\omega_0$ in order to eliminate the 60-Hz component?

e) In words, why might it be preferable to perform the signal processing in discrete time, instead of continuous time? Give a real-life example where this might apply.
(Additional space for Problem 1)
Problem 2. (Laplace Transform) Consider the causal LTI system described by the differential equation

\[ y''(t) + 6y'(t) + 9y(t) = x''(t) + 3x'(t) + 2x(t). \]

a) Find the transfer function \( H(s) \) corresponding to this system, and determine whether the system is stable.

b) Find the impulse response \( h(t) \) corresponding to this system.

c) The inverse of this system (taking \( y(t) \) back to \( x(t) \)) can also be described by a causal LTI system in differential equation form. Find the differential equation describing the inverse and its transfer function \( G(s) \).

d) What is the impulse response \( g(t) \) corresponding to the inverse system you found in part (c)?

e) Suppose a rational function \( X(s) \) has a pole-zero diagram as shown below. If you integrate the function \( X(s)e^{st} \) over the variable \( s \) along the illustrated paths (in the vertical direction indicated by the arrows), what type of time domain signal will you obtain? For each path labeled (i)-(iii), respond with ‘left-sided’, ‘right-sided’ or ‘two-sided’.

![Pole-Zero Diagram](image-url)
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Problem 3. (DFT)

a) Consider a 20-point finite duration signal $x[n]$ such that $x[n] = 0$ outside $0 \leq n \leq 19$, and let $X(e^{j\omega})$ denote the DTFT of $x[n]$. You have a computer program that can compute a DFT, but not the DTFT. However, you want to evaluate $X(e^{j\omega})$ at $\omega = 4\pi/5$. Give a method for doing so.

b) Two eight-point sequences are shown below. Signal $x_1[n]$ has 8-point DFT $X_1[k]$, and signal $x_2[n]$ has 8-point DFT $X_2[k]$. Determine the relationship between $X_1[k]$ and $X_2[k]$.

c) Suppose $x_c(t)$ is a periodic continuous-time signal with period $10^{-3}$s and for which the Fourier series is

$$x_c(t) = \sum_{k=-9}^{9} a_k e^{j2\pi kt/10^{-3}}.$$

The Fourier series coefficients are zero for $|k| > 9$. The signal $x_c(t)$ is sampled with a sampling period $T = \frac{1}{6} 10^{-3}$s to form $x[n]$. That is,

$$x[n] = x_c \left( \frac{n10^{-3}}{6} \right).$$

i) Is $x[n]$ periodic and, if so, what is the period?
ii) Is the sampling rate large enough to avoid aliasing?
iii) Compute the 6-point DFT of $\{x[0], \ldots, x[5]\}$ in terms of the $a_k$'s.
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Problem 4. (Overcoming Quantization Noise) Let $x_c(t)$ be a power signal with (continuous time) autocorrelation function

$$C_{x_c x_c}(\tau) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_c(t)x_c(t+\tau)dt = \frac{\sin(2\pi B \tau)}{\pi \tau}.$$ 

Suppose $x_c(t)$ is sampled according to $x[n] = x_c(nT_s)$, where the sampling period $T_s$ satisfies $T_s < \frac{1}{2B}$. We will assume that $x[n]$ is a power signal with

$$C_{xx}[k] = C_{x_c x_c}(kT_s). \tag{1}$$

a) Give an intuitive (non-mathematical) explanation for why (1) should hold, and compute $S_{xx}(e^{j\omega}).$

b) In practice, analog-to-digital converters cannot perfectly sample $x[n] = x_c(nT_s)$, since doing so would require an infinite number of bits. Instead, they quantize the number $x_c(nT_s)$ to an interval of length $\Delta$, which corresponds to the resolution of the analog-to-digital converter. In this manner, we model the quantized samples, denoted by $y[n]$, as

$$y[n] = x[n] + e[n],$$

where $e[n]$ is the quantization error, and is assumed to be a random number between $-\Delta/2$ and $\Delta/2$, uncorrelated from one sample to the next. Stated another way, $e[n]$ is uncorrelated with $x[n]$ and has autocorrelation function

$$C_{ee}[k] = \frac{\Delta^2}{12}\delta[k].$$

Compute $S_{xy}(e^{j\omega})$ and $S_{yy}(e^{j\omega})$.

c) Specify the optimal filter $h[n]$ (i.e., the Wiener filter) for estimating the true sample values $x[n]$ from the quantized samples $y[n]$.

d) For $\hat{x}[n] = y[n] * h[n]$, the mean square error (MSE) satisfies

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N} (x[n] - \hat{x}[n])^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( S_{xx}(e^{j\omega}) - \frac{|S_{xy}(e^{j\omega})|^2}{S_{yy}(e^{j\omega})} \right) d\omega. \tag{2}$$

Use (2) to find the MSE in estimating $x[n]$ from $y[n]$ in terms of $T_s, B$ and $\Delta$.

e) Prove that (2) holds. (Note: this relationship holds in general for the Wiener filter, and is not specific to this problem.)
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