EE 120: Signals and Systems
Department of EECS
UC BERKELEY

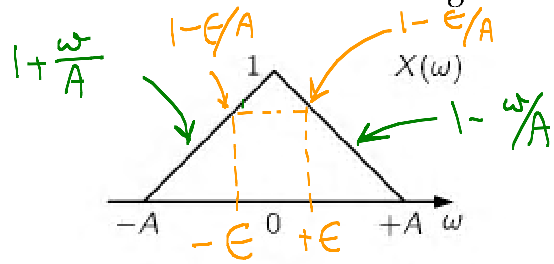
MIDTERM 2
31 March 2010

LAST Name Limited FIRST Name Tim E.

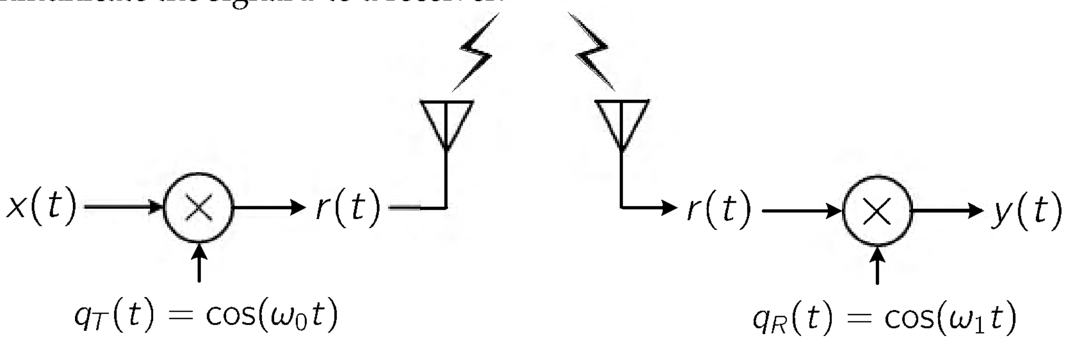
Discussion Time ?

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except **two** double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT2.1 (45 Points) A bandlimited continuous-time signal x has the triangular spectrum shown below:



The following diagram shows an amplitude modulation-demodulation scheme to communicate the signal x to a receiver:



In this problem, you'll explore the effects of frequency mismatch between the transmitter and receiver carriers q_T and q_R, respectively. In particular, assume that

$$0 < \epsilon \ll A < \omega_0 = \omega_1 + \epsilon.$$

$\omega_0 + \omega_1 = 2\omega_0 - \epsilon$

$\omega_0 - \omega_1 = \epsilon$

(a) (15 Points) Determine reasonably simple expressions for the signals r and y. You may find the following trigonometric identity useful:

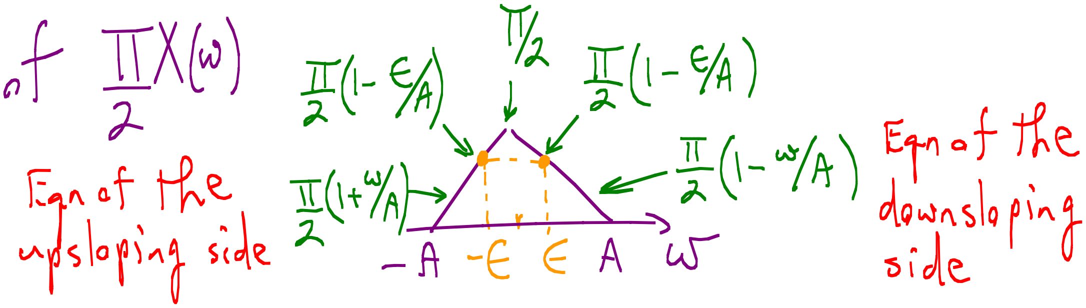
$$\cos\alpha\,\cos\beta = \frac{1}{2}\Big[\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big].$$

$r(t) = x(t)\,q(t) \implies r(t) = x(t)\cos(\omega_0 t)$

$y(t) = r(t)\,q_R(t) = x(t)\cos(\omega_0 t)\cos(\omega_1 t) = \frac{x(t)}{2}\left\{\cos\left[(\omega_0+\omega_1)t\right] + \cos\left[(\omega_0-\omega_1)t\right]\right\}$

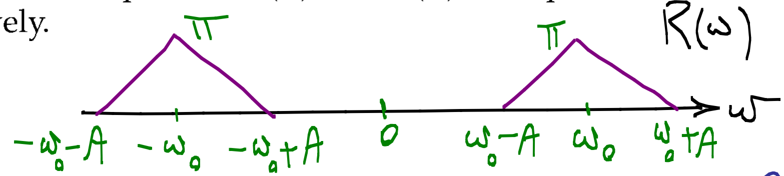
$y(t) = \frac{x(t)}{2}\left\{\cos\left[(2\omega_0-\epsilon)t\right] + \cos(\epsilon t)\right\}$

Plot of $\frac{\pi}{2}X(\omega)$

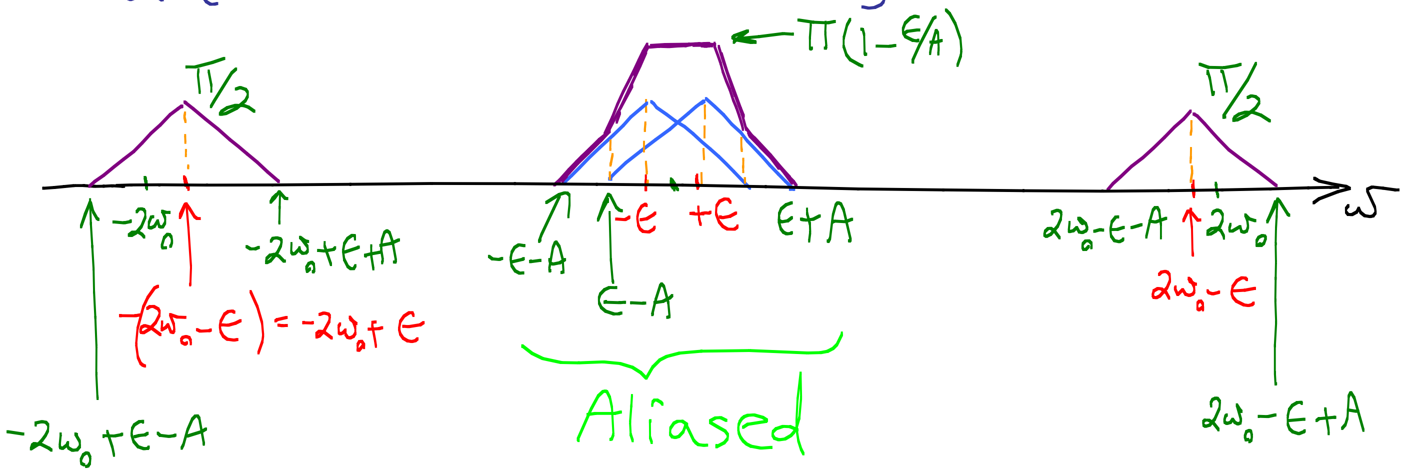


(b) (20 Points) Provide well-labeled plots of $R(\omega)$ and $Y(\omega)$, the spectra of the signals r and y, respectively.

$$R(\omega) = \pi\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$



$$Y(\omega) = \frac{\pi}{2}\left[X(\omega+(2\omega_0-\epsilon)) + X(\omega-(2\omega_0-\epsilon))\right] + \frac{\pi}{2}\left[X(\omega+\epsilon)+X(\omega-\epsilon)\right]$$



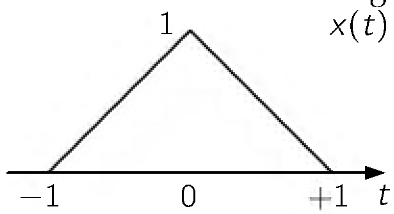
(c) (10 Points) Explain why the information-bearing signal x is irrecoverable, even if we send the signal y through an ideal low-pass filter.

Looking at the three components of the spectrum $R(\omega)$, and considering that we have no a priori knowledge of the frequency mismatch ϵ, we note that

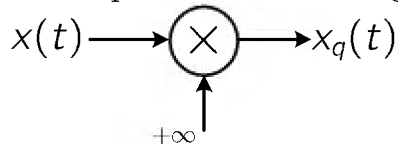
(a) low-pass filtering r doesn't recover x because of the aliasing at baseband frequencies (annotated on the diagram)

(b) no further modulation/demodulation of r will work even though at frequencies around $\pm 2\omega_0$ the triangular shape of the original signal x is intact. This is because ϵ is unknown, and hence so are the center frequencies $\pm 2\omega_0 - \epsilon$.

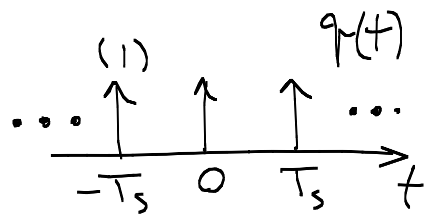
MT2.2 (60 Points) A time-limited continuous-time signal x is shown below:



We sample the signal x with an impulse train according to the figure below:



$$q(t) = \sum_{\ell=-\infty}^{+\infty} \delta(t - \ell T_s)$$



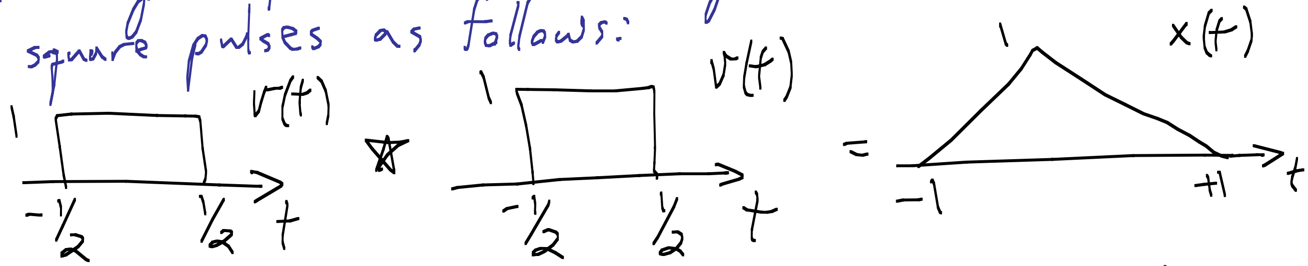
Throughout this problem, let the sampling period be $T_s = 1/N$ (seconds), where $N \in \{1, 2, 3, \ldots\}$. The corresponding sampling frequency is $\omega_s = 2\pi/T_s = 2\pi N$.

(a) (15 Points) Prove that the continuous-time Fourier transform of x is

$$\forall \omega \in \mathbb{R}, \quad X(\omega) = \frac{4}{\omega^2} \sin^2\left(\frac{\omega}{2}\right),$$

and provide a well-labeled plot of $X(\omega)$.

The triangular pulse x can be thought of the convolution of two square pulses as follows:



$x(t) = (v \star v)(t) \implies X(\omega) = V^2(\omega)$

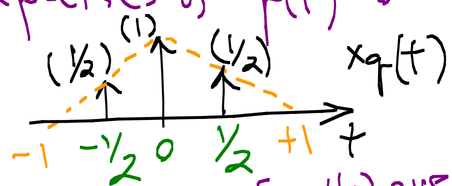
$V(\omega) = \int_{-\infty}^{\infty} v(t) e^{-i\omega t} dt = \int_{-1/2}^{1/2} e^{-i\omega t} dt = \frac{e^{-i\omega t}}{-i\omega}\Big|_{-1/2}^{1/2} = \frac{e^{i\omega/2} - e^{-i\omega/2}}{i\omega} \implies$

$V(\omega) = \frac{2}{\omega} \sin(\omega/2) \implies \underline{X(\omega) = \frac{4}{\omega^2} \sin^2(\omega/2)}$

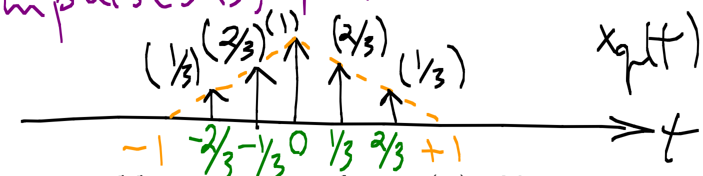
(b) (12 Points) Provide well-labeled plots of the signal x_q for each of the following cases: $N = 1$, $N = 2$, and $N = 3$.

$\underline{N=1}$: Only one of the impulses of $q(t)$ overlaps with the nonzero values of $x(t)$

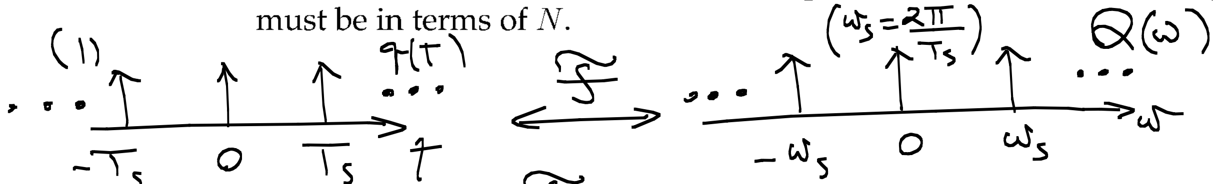
$\underline{N=2}$: Only three of the impulses of $q(t)$ overlap with the nonzero values of $x(t)$



$\underline{N=3}$: Only five of the impulses of $q(t)$ overlap with the nonzero values of $x(t)$



(c) (12 Points) Determine a reasonable expression for $X_q(\omega)$. Your expression must be in terms of N.



$$q(t) = \sum_{\ell=-\infty}^{\infty} \delta(t - \ell T_s) \xleftrightarrow{\mathcal{F}} Q(\omega) = \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$x_q(t) = x(t)\,q(t) \xleftrightarrow{\mathcal{F}} X_q(\omega) = \frac{1}{2\pi}(X * Q)(\omega) \Longrightarrow$$

$$X_q(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) = 4N \sum_{k=-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega - 2\pi k N}{2}\right)}{(\omega - 2\pi k N)^2}$$

$$\boxed{\text{Since } T_s = 1/N \Rightarrow \omega_s = 2\pi N \text{ and } X(\omega) = \frac{4}{\omega^2}\sin^2(\omega/2)}$$

$$X_q(\omega) = 4N \sum_{k=-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega}{2} - k\pi N\right)}{(\omega - 2\pi k N)^2} \Longrightarrow X_q(\omega) = 4N \sin^2(\omega/2) \sum_{k=-\infty}^{\infty} \frac{1}{(\omega - 2\pi k N)^2}$$

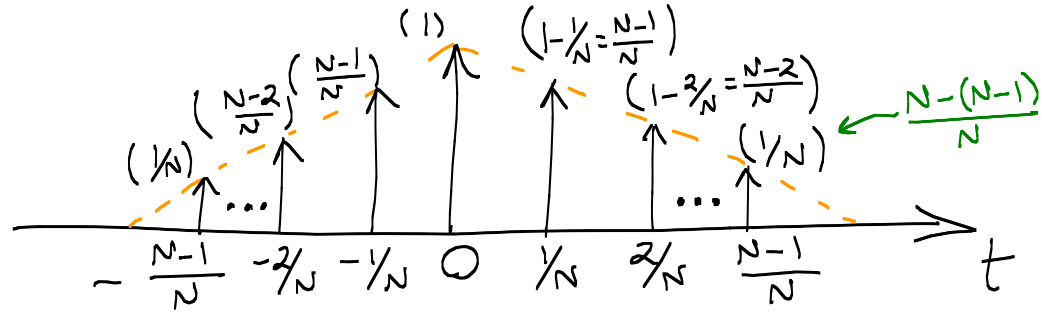
$\underline{\text{Note:}}$ $\sin\left(\frac{\omega}{2} - k\pi N\right) = (-1)^{kN} \sin(\omega/2) \Longrightarrow \sin^2\left(\frac{\omega}{2} - k\pi N\right) = \sin^2(\omega/2)$

Alternative Expressions for $x_q(t)$ and $X_q(\omega)$:

$$\boxed{x_q(t) = \delta(t) + \sum_{\ell=1}^{N-1} (1 - \ell/N)\left[\delta(t - \ell/N) + \delta(t + \ell/N)\right]}$$

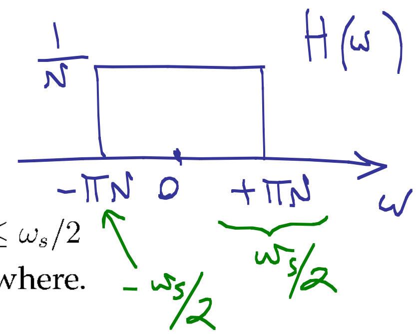
See the diagram at the top of the next page.

$$\Rightarrow X_q(\omega) = 1 + \sum_{\ell=1}^{N-1} (1 - \ell/N)\left(e^{-i\omega\ell/N} + e^{i\omega\ell/N}\right) \Rightarrow X_q(\omega) = 1 + 2\sum_{\ell=1}^{N-1} (1 - \ell/N)\cos(\omega\ell/N)$$



(d) (16 Points) We pass the signal x_q through an ideal low-pass filter H, as shown below.

$$x_q \longrightarrow \boxed{H} \longrightarrow y$$



Recall that: $T_s = \frac{1}{N} \Rightarrow \omega_s = \frac{2\pi}{T_s} = 2\pi N$

$$\Rightarrow \boxed{\frac{\omega_s}{2} = \pi N}$$

The frequency response of the filter is,

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \begin{cases} T_s = 1/N & |\omega| \le \omega_s/2 \\ 0 & \text{elsewhere.} \end{cases}$$

Determine a reasonably simple expression for each of $Y(\omega)$ and $y(t)$, the spectral and temporal representations of the filtered output signal y, respectively.

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi N}\int_{-\pi N}^{\pi N} e^{i\omega t}\,d\omega = \frac{1}{2\pi N}\,\frac{e^{i\omega t}}{it}\Bigg|_{-\pi N}^{\pi N} = \frac{e^{i\pi N t} - e^{-i\pi N t}}{2i\pi N t} = \frac{\sin(\pi N t)}{\pi N t}$$

From part (c), we know that $x_q(t) = \delta(t) + \sum_{l=1}^{N-1}\left(1 - \frac{l}{N}\right)\left[\delta\left(t - \frac{l}{N}\right) + \delta\left(t + \frac{l}{N}\right)\right] \Rightarrow$

$$y(t) = (x_q * h)(t) = h(t) + \sum_{l=1}^{N-1}(1 - l/N)\left[h(t - l/N) + h(t + l/N)\right]$$

$$= \frac{\sin(\pi N t)}{\pi N t} + \sum_{l=1}^{N-1}(1 - l/N)\left\{\frac{\sin[\pi N(t - l/N)]}{\pi N(t - l/N)} + \frac{\sin[\pi N(t + l/N)]}{\pi N(t + l/N)}\right\}$$

$$= \frac{\sin(\pi N t)}{\pi N t} + \sum_{l=1}^{N-1}(1 - l/N)\left[\frac{\sin(\pi N t - l\pi)}{\pi(Nt - l)} + \frac{\sin(\pi N t + l\pi)}{\pi(Nt + l)}\right]$$

$$= \frac{\sin(\pi N t)}{\pi N t} + \sum_{l=1}^{N-1}(1 - l/N)(-1)^l\left[\frac{1}{\pi(Nt - l)} + \frac{1}{\pi(Nt + l)}\right]\sin(\pi N t)$$

$$\boxed{y(t) = \frac{\sin(\pi N t)}{\pi N t} + \frac{2t\sin(\pi N t)}{\pi}\sum_{l=1}^{N-1}\frac{(-1)^l(N - l)}{(Nt - l)(Nt + l)}}$$

$$Y(\omega) = X_q(\omega)H(\omega) = \begin{cases} \frac{X_q(\omega)}{N} & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{e/w} \end{cases}$$

Use $X_q(\omega)$ from Part (c).

$$Y(\omega) = \begin{cases} \frac{1}{N} + \frac{2}{N}\sum_{l=1}^{N-1}\left(1 - \frac{l}{N}\right)\cos\left(\frac{\omega l}{N}\right) & \text{if } |\omega| < \frac{\omega_s}{2} = \pi N \\ 0 & \text{if } \pi N = \frac{\omega_s}{2} < |\omega| \end{cases}$$

(e) (5 Points) Why is it that no matter how finely we sample x—that is, regardless of how large N becomes, or, equivalently, how small the sampling period $T_s = 1/N$ becomes—we cannot recover x from its samples? Be succinct, but clear and convincing.

The signal x is time-limited but, as we discovered in part (a) from its spectrum $X(\omega) = \frac{4}{\omega^2}\sin^2(\omega/2)$, it's not bandlimited. Accordingly, no matter how fast we sample x, inevitably there will be aliasing. Therefore, from the samples x_q alone, we can not recover x.

$$Y(\omega) = X_q(\omega) H(\omega) = \begin{cases} \dfrac{X_q(\omega)}{N} & |\omega| < \dfrac{\omega_s}{2} = \pi N \\ 0 & \text{e/w} \end{cases}$$

A lternate approach to (f)

$$X_q(\omega) = 1 + 2\sum_{\ell=1}^{N-1}(1 - \ell/N)\cos\left(\frac{\omega\ell}{N}\right) = 1 + \frac{2}{N}\sum_{\ell=1}^{N-1}(N-\ell)\cos\left(\frac{\omega\ell}{N}\right)$$

$$H(\omega)X_q(\omega) = H(\omega) + \frac{2}{N}\sum_{\ell=1}^{N-1}(N-\ell)\cos\left(\frac{\omega\ell}{N}\right)H(\omega) = \frac{1}{N} + \frac{2}{N^2}\sum_{\ell=1}^{N-1}(N-\ell)\cos\left(\frac{\omega\ell}{N}\right) \quad |\omega| \leq \pi N$$

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

$$y(t) = h(t) + \frac{\cancel{2}}{\cancel{2}\pi N^2}\sum_{\ell=1}^{N-1}(N-\ell)\int_{-\pi N}^{\pi N}\cos\left(\frac{\omega\ell}{N}\right)e^{i\omega t}\,d\omega$$

$$\int_{-\pi N}^{\pi N}\cos\left(\frac{\omega\ell}{N}\right)\cos(\omega t)\,d\omega + i\underbrace{\int_{-\pi N}^{\pi N}\cos\left(\frac{\omega\ell}{N}\right)\sin(\omega t)\,d\omega}_{\text{odd integrand}} \rightarrow 0$$

$$\int_{-\pi N}^{\pi N}\underbrace{\cos\left(\frac{\omega\ell}{N}\right)\cos(\omega t)}_{\text{even in } \omega}\,d\omega \Rightarrow = 2\int_0^{\pi N}\cos\left(\frac{\omega\ell}{N}\right)\cos(\omega t)\,d\omega = \cancel{2}\cdot\frac{1}{\cancel{2}}\int_0^{\pi N}\left[\cos\left(\omega\left(t+\ell/N\right)\right) + \cos\left(\omega\left(t-\ell/N\right)\right)\right]d\omega$$

$$= \left.\frac{\sin[\omega(t+\ell/N)]}{t+\ell/N}\right|_0^{\pi N} + \left.\frac{\sin[\omega(t-\ell/N)]}{t-\ell/N}\right|_0^{\pi N}$$

$$= \frac{\sin[\pi(Nt+\ell)]}{t+\ell/N} + \frac{\sin[\pi(Nt-\ell)]}{t-\ell/N}$$

$$= (-1)^\ell \sin \pi N t\left(\frac{1}{t+\ell/N} + \frac{1}{t-\ell/N}\right)$$

$$= \frac{(-1)^\ell \sin(\pi N t)}{(Nt+\ell)(Nt-\ell)}(2N^2 t)$$

$$y(t) = h(t) + \frac{\sin(\pi N t)}{\pi \cancel{N^2}}\sum_{\ell=1}^{N-1}\frac{(N-\ell)(-1)^\ell\, 2\cancel{N^2}t}{(Nt+\ell)(Nt-\ell)} = h(t) + \frac{2t\sin \pi N t}{\pi}\sum_{\ell=1}^{N-1}\frac{(-1)^\ell(N-1)}{(Nt+\ell)(Nt-\ell)}$$

LAST Name Limited FIRST Name Tim E.

Discussion Time ?

Problem	Points	Your Score
Name	10	10
1(a)	15	15
1(b)	20	20
1(c)	10	10
2(a)	15	15
2(b)	12	12
2(c)	12	12
2(d)	16	16
2(e)	5	5
Total	**115**	115