EE 120: Signals and Systems
Department of EECS
UC BERKELEY

MIDTERM 3
11 May 2009

LAST Name Bode FIRST Name Mr.
Discussion Time ?

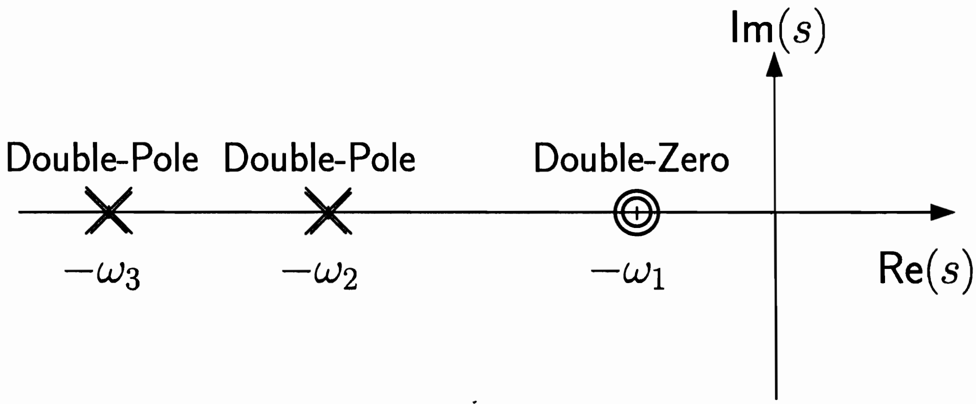
- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT3.1 (55 Points) The pole-zero diagram for a *causal* continuous-time LTI system H is shown in the figure below.

There are two zeros at $-\omega_1$, two poles at $-\omega_2$, and two poles at $-\omega_3$.

The frequencies ω_1, ω_2, and ω_3 are positive, finite, and well-separated according to the ordering $\omega_1 < \omega_2 < \omega_3$.



If the input to the system is $x(t) = 1$ for all t, the corresponding output is $y(t) = A_1$ for all t, where $0 < A_1 < \infty$.

5

(a) Must the system be BIBO stable? Explain your reasoning.

Yes, the system must be causal. We're told the system is causal, so the region of convergence of $\hat{H}(s)$, its transfer function, is to the right of the rightmost pole(s). Hence, RoC is $Re(s) > -\omega_2$ (, which includes the imaginary axis.

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(b) Suppose the input to the system is the unit-step: $x(t) = u(t)$. Does the system's response have a steady-state component y_{ss}? If you claim that it does, explain your reasoning *and* determine the steady-state response. If you claim that it does not, explain your reasoning.

$x(t) = u(t) \implies \hat{X}(s) = \frac{1}{s}$, $Re(s) > 0$. The input has a pole @ $s=0$, which is to the right of the right-most poles of the system (at $-\omega_1$). Therefore, the output does have a steady-state component. As $t \to \infty$, the response due to $x(t) = u(t)$, i.e., the steady-state component $y_{ss}(t)$, becomes indistinguishable from the response due to the input $x(t) = A_1$, $\forall t$. Therefore, $y_{ss}(t) = A_1 u(t)$ or simply $y_{ss}(t) = A_1$ t large enough.

15

(c) Determine a reasonably-simple expression for $\hat{H}(s)$, the transfer function of the system, and provide a well-labeled *Bode magnitude response* plot of H.

$$x(t) = 1 \longrightarrow \boxed{H} \longrightarrow y(t) = \hat{H}(0)\,1 \implies \hat{H}(0) = A_1 \quad \text{DC response}$$

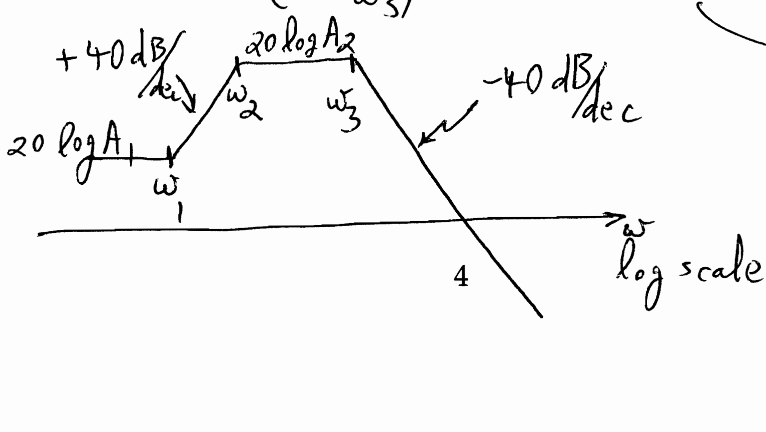
$x(t) = 1 = e^{i0t}$; frequency response $H(0)$; $1 = e^{i0t}$

Starting @ zero frequency, we begin @ $20 \log A_1$ (flat). Looking toward increasing, positive frequency direction, we encounter two regular zeros @ ω_1: $\left(1+\frac{s}{\omega_1}\right)^2$; we then see two regular poles at ω_2: $\left(1+\frac{s}{\omega_2}\right)^2$; and farther up we see another pair of poles, in particular, at ω_3: $\frac{1}{\left(1+\frac{s}{\omega_3}\right)^2}$. Therefore,

$$\hat{H}(s) = \frac{A_1\left(1+\frac{s}{\omega_1}\right)^2}{\left(1+\frac{s}{\omega_2}\right)^2\left(1+\frac{s}{\omega_3}\right)^2}$$

RoC: $Re(s) > -\omega_1$

Bode Plot:



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(d) Based on your work in part (c), determine a reasonably simple approximate expression for $|H(\omega_0)|$, the magnitude response value at frequency ω_0, where

$$\omega_0 = \sqrt{\omega_2\,\omega_3}.$$

Express your answer in terms of A_1, and one or more of the pole-zero frequencies.

On a log frequency scale, ω_0 is halfway between ω_2 and ω_3, because $\log \omega_0 = \frac{1}{2}\log(\omega_2\omega_3) = \frac{1}{2}(\log \omega_2 + \log \omega_3) \Longrightarrow |H(\omega_0)| = A_2$ (as labeled in the Bode plot of part (c)). The transfer function, written from the flatband reference $20 \log A_2$, looking down and up along the frequency axis, is $\hat{H}(s) = A_2 \dfrac{(1+\omega_1/s)^2}{(1+\frac{\omega_2}{s})^2(1+s/\omega_3)^2}$

An inverted pair of zeros @ ω_1; an inverted pair of poles @ ω_2; and a regular pair of poles @ ω_3. Rewriting $\hat{H}(s)$, we have: $\hat{H}(s) = A_2 \dfrac{(\frac{\omega_1}{\cancel{s}})^2(1+s/\omega_1)^2}{(\frac{\omega_2}{\cancel{s}})^2(1+s/\omega_2)^2(1+s/\omega_3)^2}$

So, we must have: $A_1 = A_2\left(\frac{\omega_1}{\omega_2}\right)^2 \Longrightarrow \underline{|H(\omega_0)| \approx A_2 = A_1\left(\frac{\omega_2}{\omega_1}\right)^2}$

(e) The expressions below are offered as candidate impulse responses for the system; the coefficients are all real and finite.

10

$$\begin{aligned}
h_I(t) &= \alpha_1\, e^{-\omega_1 t}\, u(t) + \alpha_2\, t\, e^{-\omega_1 t}\, u(t)\\
h_{II}(t) &= \beta_1\, e^{-\omega_2 t}\, u(t) + \gamma_1\, e^{-\omega_3 t}\, u(t)\\
\longrightarrow \boxed{h_{III}(t)} &\boxed{= \beta_1\, e^{-\omega_2 t}\, u(t) + \beta_2\, t\, e^{-\omega_2 t}\, u(t) + \gamma_1\, e^{-\omega_3 t}\, u(t) + \gamma_2\, t\, e^{-\omega_3 t}\, u(t)}\\
h_{IV}(t) &= \alpha_1\, e^{-\omega_1 t}\, u(t) + \alpha_2\, t\, e^{-\omega_1 t}\, u(t) + \beta_1\, e^{-\omega_2 t}\, u(t) + \beta_2\, t\, e^{-\omega_2 t}\, u(t)\\
&\quad + \gamma_1\, e^{-\omega_3 t}\, u(t) + \gamma_2\, t\, e^{-\omega_3 t}\, u(t)
\end{aligned}$$

From the list above, choose every valid candidate impulse response for the system. Explain your reasoning succinctly, but clearly and convincingly.

The partial fraction expansion of $\hat{H}(s)$ will have the following terms:

$$\hat{H}(s) = \frac{\hat{\beta}_1}{(1+s/\omega_2)} + \frac{\hat{\beta}_2}{(1+\frac{s}{\omega_2})^2} + \frac{\hat{\gamma}_1}{(1+\frac{s}{\omega_3})} + \frac{\hat{\gamma}_2}{(1+\frac{s}{\omega_3})^2} = \frac{\beta_1}{(s+\omega_2)} + \frac{\beta_2}{(s+\omega_2)^2} + \frac{\gamma_1}{(s+\omega_3)} + \frac{\gamma_2}{(s+\omega_3)^2},$$

where $\beta_1 = \omega_2 \hat{\beta}_1$, $\beta_2 = \omega_2^2 \hat{\beta}_2$, $\gamma_1 = \omega_3 \hat{\gamma}_1$, and $\gamma_2 = \omega_3^2 \hat{\gamma}_2$ are all nonzero.

The inverse transform, then, is

$$h_{\underline{III}}(t) = \beta_1 e^{-\omega_2 t} u(t) + \beta_2 t e^{-\omega_2 t} u(t) + \gamma_1 e^{-\omega_3 t} u(t) + \gamma_2 t e^{-\omega_3 t} u(t)$$

MT3.2 (20 Points) Let g_0 be a finite-duration continuous-time signal, such that

$$g_0(t) = 0, \quad \text{for } t < 0 \text{ and } 0 < T < t.$$

We construct a related signal g as follows:

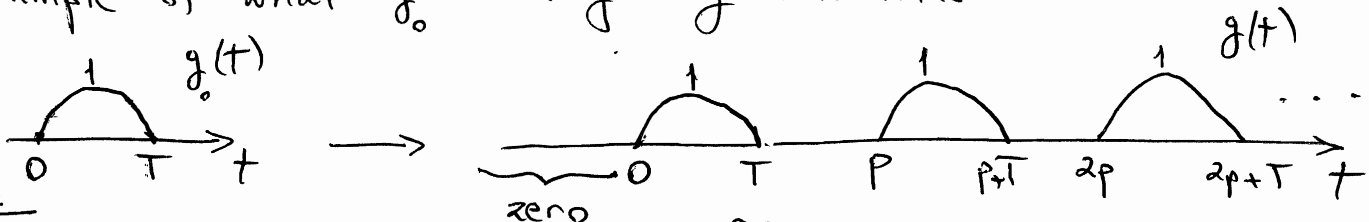
$$g(t) = \sum_{\ell=0}^{+\infty} g_0(t - \ell p), \quad \forall t,$$

where $T < p < \infty$.

Let $\widehat{G}_0(s)$ represent the Laplace transform of the finite-duration signal g_0.

Determine $\widehat{G}(s)$, the Laplace transform of g. Your answer should be in terms of p and $\widehat{G}_0(s)$.

Sample of what g_0 and g may look like:



Method 1:

$$\hat{G}(s) = \int_0^\infty g(t) e^{-st} dt = \underbrace{\int_0^p g(t) e^{-st} dt}_{} + \int_p^\infty g(t) e^{-st} dt \qquad \text{let } t = \tau + p \Rightarrow dt = d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} g_0(t) e^{-st} dt} + \int_0^\infty \underbrace{g(\tau + p)}_{g(\tau)} e^{-s(\tau+p)} d\tau$$

$$= \hat{G}_0(s) + e^{-sp} \underbrace{\int_0^\infty g(\tau) e^{-s\tau} d\tau}_{\hat{G}(s)} \Rightarrow$$

$$\hat{G}(s) = \hat{G}_0 + e^{-sp} \hat{G}(s) \Rightarrow$$

$$\hat{G}(s) = \frac{\hat{G}_0(s)}{1 - e^{-sp}}$$

Method 2: $g(t) = \sum_{\ell=0}^{\infty} g_0(t - \ell p) \Rightarrow \quad \hat{G}(s) = \sum_{\ell=0}^{\infty} \hat{G}_0(s) e^{-sp\ell} = \hat{G}_0(s) \sum_{\ell=0}^{\infty} (e^{-sp})^\ell$

If $|e^{-sp}| < 1$ (i.e., if $Re(s) > 0$), then $\sum_{\ell=0}^{\infty} (e^{-sp})^\ell = \frac{1}{1 - e^{-sp}} \Rightarrow \hat{G}(s) = \frac{\hat{G}_0(s)}{1 - e^{-sp}}$

MT3.3 (30 Points) [Initial Value Theorem] Consider a *causal* continuous-time signal x. Assume x has no infinite discontinuities at $t = 0$. In this problem, you will prove the initial value theorem:

$$x(0^+) = \lim_{s\to\infty} s\,\widehat{X}(s),$$

where $\widehat{X}$ denotes the Laplace transform of x, and $x(0^+)$ is the initial value of x (immediately after $t = 0$). We can expand x as a Taylor series about $t = 0$ as follows:

$$x(t) = \left[x(0^+) + x^{(1)}(0^+)\,t + \cdots + x^{(n)}(0^+)\,\frac{t^n}{n!} + \cdots\right] u(t),$$

where $x^{(n)}(0^+)$ denotes the nth derivative of x at $t = 0^+$.

(a) Show that $1/s^{n+1}$ is the Laplace transform of $\frac{t^n}{n!}u(t)$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \;; \quad t\,u(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds}\hat{U}(s) = \frac{1}{s^2} \;; \quad t^2 u(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3} \;; \cdots$$

$$t^n u(t) \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}} \implies \frac{t^n}{n!}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^{n+1}}$$

(b) Show that

$$\widehat{X}(s) = \sum_{n=0}^{+\infty} x^{(n)}(0^+)\frac{1}{s^{n+1}},$$

and hence prove the initial value theorem.

Taking a term-by-term Laplace transform of the Taylor expansion for x, we have:

$$\hat{X}(s) = \sum_{n=0}^{\infty} \mathcal{L}\left[x^{(n)}(0^+)\frac{t^n}{n!}u(t)\right] = \sum_{n=0}^{\infty} x^{(n)}(0^+)\,\mathcal{L}\left[\frac{t^n}{n!}u(t)\right] = \sum_{n=0}^{\infty} \frac{x^{(n)}(0^+)}{s^{n+1}} \implies$$

$$s\hat{X}(s) = s\left[\frac{x(0^+)}{s} + \frac{x^{(1)}(0^+)}{s^2} + \cdots + \frac{x^{(n)}(0^+)}{s^{n+1}} + \cdots\right] = x(0^+) + \frac{x^{(1)}(0^+)}{s} + \cdots + \frac{x^{(n)}(0^+)}{s^n} + \cdots$$

$$\lim_{s\to\infty} s\hat{X}(s) = x(0^+)$$ only the first term may be nonzero

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Problem Name	Points	Your Score
	10	10
1	55	55
2	20	20
3	30	30
Total	**115**	115