EE 120: Signals and Systems 19 February 2008
Department of EECS El Ghaoui
UC BERKELEY

Midterm 1

LAST Name ________________________ FIRST Name ________________________

Student ID _______________

- **(10 Points)** Print your name and student ID in legible, block lettering above AND on the last page where the grading table appears.
- You will be given up to a maximum of 110 minutes to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5”×11” sheet of handwritten notes having no appendage. Computing, communication and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff – *including, for example, commencing work prematurely or continuing beyond the announced stop time* – is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **This exam printout consists of pages numbered 1 through 11.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can’t read it, we can’t grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (20 Points) Consider a causal LTI system H described by the difference equation

$$y(n) + \frac{1}{2} y(n-1) = x(n) - \frac{1}{2} x(n-1).$$

(a) Determine the impulse response $h(n)$ of this system.

(b) Determine the frequency response $H(\omega)$ of this system.

(c) Determine the response of this system to the following inputs.

(i) $x(n) = (\tfrac{1}{2})^n u(n)$, where $u(n)$ is the unit step function.

(ii) $x(n) = \delta(n-2) + \dfrac{1}{2}\delta(n-3)$.

(iii) $x(n) = \cos(\pi n) + i^n$, where $i = \sqrt{-1}$.

MT1.2 (10 Points) Consider a discrete-time signal $x : Z \to R$ that satisfies:

I. For every $n, l \in Z$, $x(n+4l) = x(n)$.

II. $\sum_{n=-1}^{2} x(n) = 2$.

III. $\sum_{n=-1}^{2} (-1)^n x(n) = 1$.

IV. $\sum_{n=-1}^{2} x(n)\cos(n\pi/2) = \sum_{n=-1}^{2} x(n)\sin(n\pi/2) = 0$.

(a) Determine the complex exponential Fourier series coefficients X_{-1}, X_0, X_1 and X_2 for this signal.

(b) Determine an expression for the signal x itself.

MT1.3 (20 Points) Consider two discrete-time systems S_1 and S_2 that are described as follows. System S_1 has state s_1, inputs x_1, x_2 and output y and the following state-space model:

$$(S_1): \quad s_1(n+1) = As_1(n) + B_1x_1(n) + B_2x_2(n),\ y(n) = Cs_1(n),$$

and system S_2 has state s_2, input w and output p and the following state-space model:

$$(S_2): \quad s_2(n+1) = Fs_2(n) + Gw(n),\ p(n) = Hs_2(n) + Jw(n),$$

In the models above, we do not specify the size of the matrices involved, but you can assume that they all have the correct dimensions (for example, G has row size given by the dimension of the state vector s_2).

(a) Suppose that the causal LTI system S_2 is characterized by the following linear, constant-coefficient difference equation (LCCDE):

$$p(n) + p(n-2) = 2w(n) - w(n-1).$$

Construct the state-space model for S_2 by finding the matrices F, G, H and J.

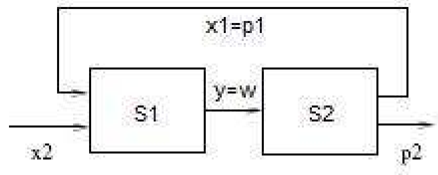

Figure 1: A connected system.

Now consider the block diagram shown in Figure 1, where $x_1 = p_1$ and $w = y$. Once again, system S_1 has state s_1, inputs x_1, x_2 and output y and the following state-space model:

$$(S_1): \quad s_1(n+1) = As_1(n) + B_1x_1(n) + B_2x_2(n),\ y(n) = Cs_1(n).$$

However, system S_2 now has state s_2, input w and **outputs p_1 and p_2** and the following state-space model:

$$(S_2): \quad s_2(n+1) = Fs_2(n) + Gw(n),\ p_1(n) = H_1s_2(n),\ p_2(n) = H_2s_2(n) + Jw(n),$$

For part (b), ignore the LCCDE representation and state-space model from part (a) completely – just leave your answer in terms of F, G, H_1, H_2 and J.

(b) Find a state-space model for this connected system, making sure to specify the state, inputs and outputs and all the system matrices. In other words, explicitly specify $\tilde{s}$, $\tilde{x}$ and $\tilde{y}$ as well as $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ satisfying

$$\tilde{s}(n+1) = A\tilde{s}(n) + B\tilde{x}(n),\ \tilde{y}(n) = C\tilde{s}(n) + D\tilde{x}(n).$$

MT1.4 (10 Points) Suppose that x and y are scalar-valued discrete-time signals (i.e., sequences) related via a (LTI) convolution system:

$$y(n) = \sum_{m} h(m)x(n-m), \; n \in Z,$$

where $\forall m, h(m) \in R$. You may assume that the system is causal.

(a) Assume that $\forall n < 0, x(n) = 0$, and let $X = \big(x(0), \ldots, x(N)\big)$ and $Y = \big(y(0), \ldots, y(N)\big)$ be two vectors of length $N + 1$ that give the first $N + 1$ values of the input and output, respectively. Express the relation between X and Y in the form of $Y = TX$, where T is a matrix you will determine in terms of h.

(b) Assume now that $x(n) = 0$ for $n < -N$ or $n > 0$. Define $Y = \big(y(0), \ldots, y(N)\big)$ and $X = \big(x(-N), \ldots, x(0)\big)$. Find a matrix H such that $Y = HX$.

MT1.5 (30 Points) Consider the equations

$$s_k(n+1) = a_k s_k(n) + b_k x(n),\ n \ge 0\,.$$

This system is a simplistic model of the dynamics of prices of N financial assets, with $s_k(n)$ the rate of return of asset k at time n, and with x a signal that represents a common factor (say, the price of oil) affecting the entire market. In the equations above, $a_k \in R$ and $b_k \in R$ are given and are independent of time.

A given investment strategy is represented by a sequence w of vectors $w(n) \in R^N$, with $w_k(n)$ the relative amount invested in asset k at time n. We assume that $\forall n \ge 0$, $w_k(n) \ge 0$ for every $k = 1, \dots, N$ (we buy but do not sell), and that $w_1(n) + \cdots + w_N(n) = 1$ (so that $w_k(n)$ represents the proportion of wealth invested in asset k at time n). The quantity $y(n) := w(n)^T s(n)$ represents the rate of return of the portfolio at time n. In this problem, we focus on the resulting SISO system, which has x as input and y as output.

(a) Devise a state-space model for this SISO system, making sure to specify the state and all the system matrices.

(b) Determine if the system has the following properties. For each property, prove that it holds (irrespective of the choice of the investment strategy w), or give a counterexample of a specific strategy that violates the property. You should assume that w is given a priori and not a function of the input.

(i) *Causal:*

(ii) *Linear:*

(iii) *Time-invariant:*

From now on, we focus on a "buy-and-hold" strategy, where $w(n)$ is independent of n and is denoted simply by $w \in R^N$, with $w_k \geq 0$ for every $k = 1, \ldots, N$ and $w_1 + \cdots + w_N = 1$.

(c) Determine the impulse response h of the system.

(d) Determine the frequency response H of the system.

(e) Determine the step response y_u of the system.

(f) Select the strongest assertion, which is true, from the choices below. Explain your choice succinctly, but clearly and convincingly.

(i) The system is BIBO stable if $|a_k| < 1$ for every $k = 1,\dots,N$.

(ii) The system is BIBO stable if and only if $|a_k| < 1$ for every $k = 1,\dots,N$.

(iii) The system is BIBO stable if and only if for every $k = 1,\dots,N$, $|a_k| < 1$ or $b_k = 0$.

Now assume that $|b_k| = 1$, $|a_k| < 1$ for every $k = 1,\dots,N$, and we define $\overline{a} = \max_{1\le k\le N}|a_k|$.

(g) Show that if $\overline{a} \le \dfrac{2}{3}$, then for every sequence x such that $x(n) \in [-0.1, 0.1]$ for every n, we have $y(n) \in [-0.3, 0.3]$ for every n.

LAST Name ________________________ FIRST Name ________________________

Student ID _______________

Problem Name	**Points**	**Your Score**
Name	10	10
1	20	
2	10	
3	20	
4	10	
5	30	
Total	100	