EECS 120, Midterm 1 Solution, 3/8/06

Do your calculations on the sheets and put a box around your answer where this makes sense. Print your name and your TA's name and section time here:

Last Name ____________ First ____________ TA's name ____________ Section time ____________

Prob #	Max	Score
1	15	
2	15	
3	10	
4	15	
5	20	
6	10	
7	15	
Total	100	

Useful tips

FT pairs

Signal $t \to x(t)$	$\omega \to X(\omega)$	$f \to X(f)$
$X(t)$	$2\pi x(-\omega)$	$x(-f)$
$x(t) \equiv 1$	$X(\omega) = 2\pi\delta(\omega)$	$X(f) = \delta(f)$
$x(t) = \delta(t)$	$X(\omega) \equiv 1$	$X(f) \equiv 1$
$x(t) = \text{sgn}(t)$	$X(\omega) = \frac{2}{j\omega}$	$X(f) = \frac{1}{j\pi f}$
$x(t) = u(t)$	$X(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$	$X(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$
$x(t) = \frac{1}{\pi t}$	$X(\omega) = -j\text{sgn}(\omega)$	$X(f) = -j\text{sgn}(f)$
$x(t) = \Pi(t)$ =1, $\lvert t\rvert \leq 1/2$, 0 else	$\frac{\sin(\pi f)}{\pi f}$	$\frac{\sin(\omega/2)}{\omega/2}$
$\sum \delta(t - kT_0)$	$\frac{1}{T_0}\sum \delta(f - f_0)$	$\frac{2\pi}{T_0}\sum \delta(\omega - \omega_0)$
$X_n \sum e^{jn\omega_0 t}$	$\sum X_n \delta(f - nf_0)$	$2\pi \sum \delta(\omega - n\omega_0)$
$\hat{x}(t)$	$-j\text{sgn}(f)X(f)$	$-j\text{sgn}(\omega)X(\omega)$

FT properties

$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{f}{a})$	$\frac{1}{\lvert a\rvert}X(\frac{\omega}{a})$
$x * y$	$X(f)Y(f)$	$X(\omega)Y(\omega)$
$x(t)y(t)$	$(X * Y)(f)$	$\frac{1}{2\pi}(X * Y)(\omega)$
$X(t)$	$x(-f)$	$2\pi x(-\omega)$
$e^{2\pi f_0 t}x(t)$	$X(f - f_0)$	$X(\omega - \omega_0)$
$\dot{x}(t)$	$(j2\pi f)X(f)$	$(j\omega)X(\omega)$
$\int_{-\infty}^{t} x(s)ds$	$\frac{1}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$	$\frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$

Parseval's theorem.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Trig identities

$$\begin{aligned}
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\sin x \sin y &= \frac{1}{2}[\cos(x - y) - \cos(x + y)] \\
\cos x \cos y &= \frac{1}{2}[\cos(x - y) + \cos(x + y)] \\
\sin x \cos y &= \frac{1}{2}[\sin(x - y) + \sin(x + y)
\end{aligned}$$

1. **15 points** Find the FT of x, and sketch the real and imaginary parts of $X(\omega)$, where

$$\forall t, \quad x(t) = \Pi(t) * \Pi(t) * \sum_{-\infty}^{\infty} \delta(t - 8n).$$

Here $\Pi(t) = 1$ for $|t| \le 1/2$ and 0, else. In your sketch carefully mark the relevant frequencies and magnitudes.

Answer

From Table above the FT of

$$\Pi \leftrightarrow X_1 : \omega \mapsto \frac{\sin(\omega/2)}{\omega/2} .$$

From the Table above,

$$t \mapsto \sum_{-\infty}^{\infty} \delta(t - 8n) \leftrightarrow X_2 : \omega \mapsto \frac{2\pi}{8} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{8}).$$

By the convolution property,

$$\begin{aligned} x \leftrightarrow X(\omega) &= [X_1(\omega)]^2 X_2(\omega) \\ &= \left[\frac{\sin(\omega/2)}{\omega/2}\right]^2 \frac{2\pi}{8} \sum_k \delta(\omega - \frac{2\pi k}{8}) \\ &= \frac{2\pi}{8} \sum_k \left[\frac{\sin(2\pi k/16)}{2\pi k/16}\right]^2 \delta(\omega - \frac{2\pi k}{8}) \end{aligned}$$

X is a real-valued function, its imaginary part is zero. A sketch of X is shown in figure 1.

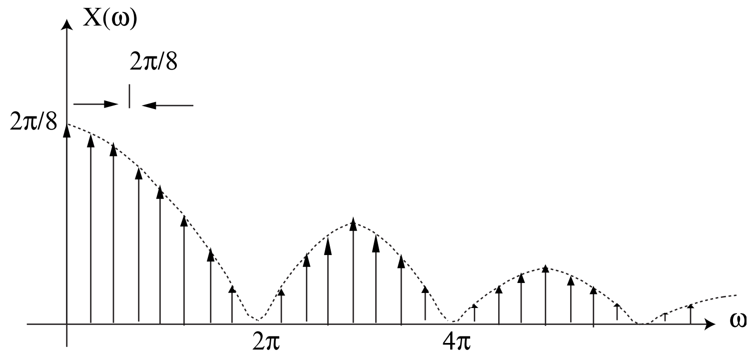

Figure 1: Sketch of X in problem 1. Only the positive frequencies are shown, since $X(\omega) = X(-\omega)$.

2. **15 points**

   (a) Find and sketch the FT of
$$x(t) = \left[\frac{\sin \pi t}{\pi t}\right]^2 e^{-j2\pi\times 10t}.$$

   (b) Use Parseval's theorem to find the energy in the signal x.

Answer

From Table above,
$$\Pi(t) \leftrightarrow \frac{\sin \omega/2}{\omega/2}\ .$$
From Table above,
$$\frac{\sin \pi t}{\pi t} \leftrightarrow 2\pi\Pi(-\omega) = 2\pi\Pi(\omega)$$
From Table above, $[x(t)]^2 \leftrightarrow (1/2\pi)(X * X)(\omega)$, so
$$\left[\frac{\sin \pi t}{\pi t}\right]^2 \leftrightarrow \frac{1}{2\pi}(2\pi)^2(\Pi * \Pi)(\omega) = 2\pi(\Pi * \Pi)(\omega)$$
$\Pi * \Pi$ has the triangle shaped graph shown in Figure 2.

   (a) The FT of x is
$$X(\omega) = 2\pi(\Pi * \Pi)(\omega - 2\pi \times 10)$$
   and is sketched in Figure 2.

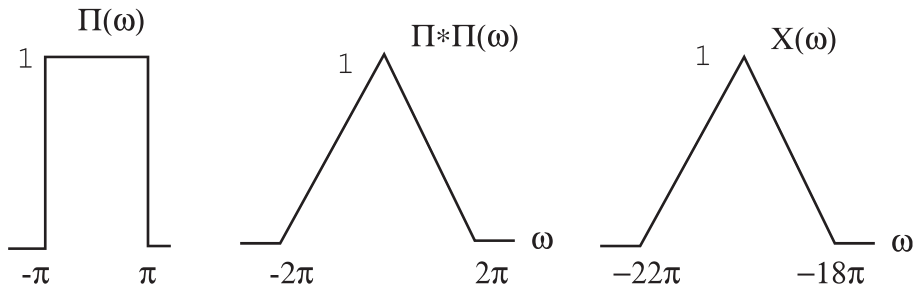

Figure 2: Sketch of X for problem 2

   (b) By Parseval's theorem the energy in x is
$$\int |x(t)|^2 dt = \frac{1}{2\pi}\int |X(\omega)|^2 d\omega = \frac{2}{2\pi}\int_0^{2\pi}\left[\frac{\omega}{2\pi}\right]^2 d\omega = \boxed{\tfrac{2}{3}}$$

3. **10 points** The following statements are either TRUE or FALSE. If you believe a statement is true, outline a BRIEF PROOF. If you believe it is false, provide a BRIEF COUNTEREXAMPLE.

   (a) If $x(t), t \in$ *Reals*, is a real-valued signal, its Fourier transform $X(f), f \in$ *Reals*, is also real-valued.

   (b) If $x(t), y(t), t \in$ *Reals*, are real-valued signals and $(x * y)(t) = 0, \forall t \in$ *Reals*, then either x or y is identically zero.

   (c) If $x(t), t \in$ *Reals*, is a real-valued, baseband signal with bandwidth W Hz, then the signal y, $y(t) = x^4(t), t \in$ *Reals*, has bandwidth at most $4W$ Hz.

   (d) If $x(t), t \in$ *Reals* is a real-valued, band-limited signal with bandwidth W Hz, then the signal $y(t) = x(2t), t \in$ *Reals*, has bandwidth W^2 Hz.

   (e) If x, y are real-valued signals with bandwidth W_x, W_y Hz, respectively, then the signal $x + y$ has bandwidth $W_x + W_y$ Hz.

Answer

(a) FALSE. The function $\forall t, x(t) = \text{sgn}(t)$ is real-valued, but its Fourier transform is $\forall \omega, X(\omega) = \frac{2}{j\omega}$ which is not real-valued.

(b) FALSE. Take a band-limited signal x for example, $x(t) = \sin(t)/t$ has bandwidth 1 radian/sec, so $|X(\omega)| = 0, |\omega| > 1$. Now take $y(t) = x(t)\cos(10t)$. Then $Y(\omega) = 1/2[X(\omega - 10) + X(\omega + 10)$. It follows that $X(\omega)Y(\omega) = 0$ for all ω. But then $(x * y)(t) = 0$ for all t, even though neither x nor y is identically zero.

(c) TRUE. $y \leftrightarrow Y = X * X * X * X$. If $X(f) = 0$ for $|f| > W$, then $X * X * X * X(f) = 0$ for $|f| > 4W$.

(d) FALSE. From the time scale property, $Y(f) = 1/2X(f/2)$. So the bandwidth of y is $2W \neq W^2$, if $W \neq 2$.

(e) FALSE. Take $y = x$. Then $x + y \leftrightarrow 2X$ has bandwidth W_x.

4. **15 points** The following statements are either TRUE or FALSE. If you believe a statement is true, outline a BRIEF PROOF. If you believe it is false, provide a BRIEF COUNTEREXAMPLE.

   (a) The system that takes as input a signal m and produces its Hilbert transform $\hat{m}$ as output is an LTI system.

   (b) The SSB-USB modulator which takes as input a signal $m(t), t \in$ *Reals*, and produces as output the modulated signal $x(t), t \in$ *Reals*, is a linear system.

   (c) The narrow-band FM system which takes as input the continuous-time signal m and produces as output the modulated signal x, is a linear system.

   (d) The AM-DSB modulator is a time-invariant system.

   (e) The signal $\forall t, x(t) = \cos(2\pi f_c t + \cos(2\pi f_m t))$ has infinite bandwidth.

   (f) It is possible to recover the signals A and θ from the narrowband signal $\forall t, x(t) = A(t)\cos(2\pi f_c t + \theta(t))$.

   Answer

   (a) TRUE, because $\hat{m} = m * h$ where h is the impulse response of the Hilbert transform.

   (b) TRUE, because

   $$\begin{aligned} \forall t, \quad x(t) &= [m(t) + j\hat{m}(t)]e^{j\omega_c t}, \\ &= (m * (\delta + jh))(t)e^{j\omega_c t}, \end{aligned}$$

   (where h is as in part (a)) and the operations above are linear.

   (c) FALSE, because the signal x is

   $$\forall t, \quad x(t) = \cos 2\pi f_c t - m(t)\sin 2\pi f_c t,$$

   and this is not a linear relation: for example $x \neq 0$ even if $m(t) \equiv 0$.

   (d) FALSE. The modulator is a linear, memoryless, *time-varying* system:

   $$x(t) = m(t)\cos 2\pi f_c t.$$

   (e) TRUE. The signal is periodic in t with period $1/f_m$ since $x(t + 1/f_m) = \cos(\cos(2\pi f_m(t + 1/f_m)) = \cos(\cos(2\pi f_m t))$. It is also an even function of t. Hence x has a Fourier series representation

   $$x(t) = \sum_{k=0}^{\infty} a_k \cos(k2\pi f_m t),$$

   which has infinite bandwidth.

   (f) TRUE. Construct the signal z by

   $$\forall t, \quad z(t) = [x(t) + j\hat{x}(t)]e^{-j2\pi f_c t},$$

   where $\hat{x}$ is the Hilbert transform of x. Then $z(t) = A(t)e^{j\theta(t)}$, so $A(t) = |z(t)|$ and $\theta(t) = \angle z(t)$.

5. **20 points** Figure 3 is a block diagram of vestigial sideband (VSB) modulation/demodulation. The

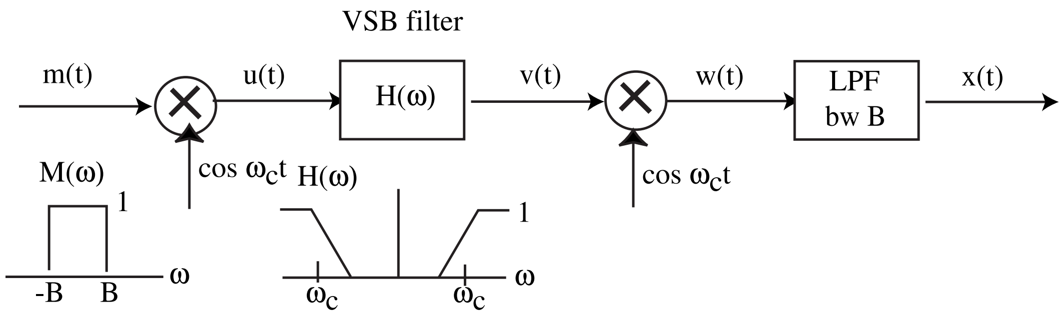

Figure 3: The VSB modulation-demodulation scheme of problem 5

baseband signal m has FT M as shown, with bandwidth B rad/sec. It modulates the carrier $\cos(\omega_c t)$ ($\omega_c >> B$) to produce the signal u, which is passed through the VSB filter, whose frequency response $H(\omega)$ is shown. The result is the transmitted signal v. The coherent receiver multiplies v by the carrier to produce w, which is then passed through a low pass filter (LPF) to obtain the signal x.

(a) Sketch the FT of u, v, and w. Carefully mark relevant magnitudes and frequencies.

(b) Show that $x = (1/4)m$ if the VSB filter satisfies

$$H(\omega + \omega_c) + H(\omega - \omega_c) = 1, \text{ for } |\omega| \le B.$$

Answer We have

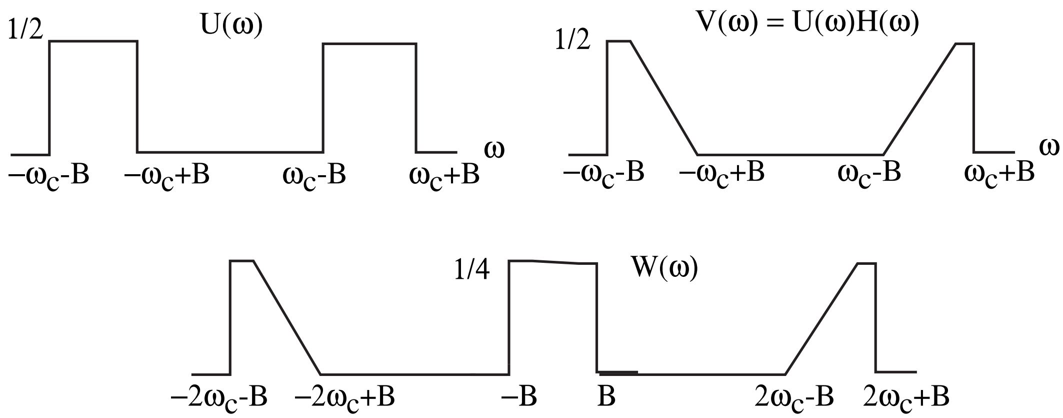

Figure 4: The FTs for problem 5

$$\begin{aligned}
U(\omega) &= \frac{1}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)], \quad V(\omega) = \frac{1}{2}H(\omega)[M(\omega - \omega_c) + M(\omega + \omega_c)] \\
W(\omega) &= \frac{1}{2}[V(\omega - \omega_c) + V(\omega + \omega_c)] \\
&= \frac{1}{4}\Big[H(\omega - \omega_c)\{M(\omega - 2\omega_c) + M(\omega)\} + H(\omega - \omega_c)\{M(\omega) + M(\omega + 2\omega_c\}\Big] \\
&= \frac{1}{4}M(\omega)[H(\omega - \omega_c) + H(\omega + \omega_c)], \text{ for } |\omega| < B
\end{aligned}$$

Figure 4 shows the FTs for (a). Part (b) follows from the last equation.

6. **10 points** Figure 5 is a block diagram of a digital communication system. The digital channel accepts

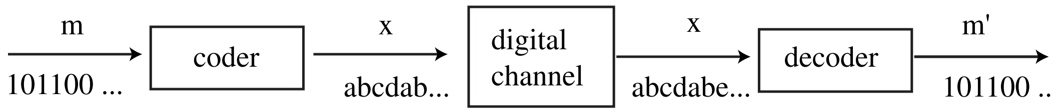

Figure 5: The communication system for problem 5

at its input port any symbol from $\{a, b, c, d, e\}$ and delivers it at its output port. The channel can accept one symbol every 2μsec.

(a) What is the baud rate of the channel? What is its capacity in bits/sec?

(b) A binary source m produces data at 1Mb/sec. (1 Mb is one million bits.) Is the rate of the source smaller than the capacity? If it is, construct a "coder" that maps the binary source m into a sequence of symbols x, and a "decoder" that maps x into a binary sequence m' such that $m' = m$.

Answer

(a) The baud rate is $1/(2 \times 10^{-6}) = 500,000$ symbols/sec. The channel capacity is $C = \log_2(5) \times 500,000$ bps.

(b) Yes. $C > 10^6$, since $\log_2(5) > \log_2 4 = 2$.
The coder should map pairs of bits into distinct symbols, the decoder should do the inverse. They are given by the following assignments:

$$\text{Coder}: 00 \rightarrow a; 01 \rightarrow b; 10 \rightarrow c; 11 \rightarrow d$$
$$\text{Decoder}: a \rightarrow 00; b \rightarrow 01; c \rightarrow 10; d \rightarrow 11$$

7. **20 points** m is a **complex-valued** signal with bandwidth B_m rad/sec whose real and imaginary parts are m_1, m_2 respectively. Let $M(\omega), M_1(\omega)$ and $M_2(\omega)$ be the FT of m, m_1 and m_2, respectively.

   (a) Find M_1 and M_2 in terms of M. Show that the bandwidth of m_1, m_2 is at most B_m.

   (b) Design a modulation and demodulation scheme that can transmit m_1 and m_2 over a channel with bandwidth $2B_m$ centered at frequency ω_c rad/sec.

   (c) Give a brief mathematical argument to show that the transmitted signal is within the channel bandwidth, and that the receiver can recover both signals.

Answer

(a) $M(\omega) = \int m(t)e^{-j\omega t}dt;\ M^*(\omega) = \int m^*(t)e^{j\omega t}dt;\ M^*(-\omega) = \int m^*(t)e^{-j\omega t}dt$. Since $m + m^* = 2m_1$ and $m - m^* = 2jm_2$,

$$M(\omega) + M^*(-\omega) = \int [m(t) + m^*(t)]e^{-j\omega t}dt = 2M_1(\omega)$$
$$M(\omega) - M^*(-\omega) = \int [m(t) - m^*(t)]e^{-j\omega t}dt = 2jM_2(\omega)$$

(b) The modulated signal is

$$x(t) = m_1(t)\cos(\omega_c t) + m_2(t)\sin(\omega_c t).$$

To recover the signals we use coherent demodulation. Multiply x by $\cos\omega_c t$ and pass the product through a LPF with cutoff B_m to recover m_1; multiply x by $\sin\omega_c t$ and pass the product through a LPF with cutoff B_m Hz to recover m_2.

(c) We have

$$\begin{aligned} x(t) &\leftrightarrow X(\omega) \\ &= M_1(\omega) * \frac{1}{2}[\delta(\omega-\omega_c) + \delta(\omega+\omega_c)] + M_2(\omega)\frac{1}{2j}[\delta(\omega-\omega_c) - \delta(\omega+\omega_c)] \\ &= \frac{1}{2}[M_1(\omega-\omega_c) + M_1(\omega+\omega_c)] + \frac{1}{2j}[M_2(\omega-\omega_c) - M_2(\omega+\omega_c)], \end{aligned}$$

which shows that $|X(\omega)| = 0, ||\omega| - \omega_c| > B_m$.
To show that the demodulation scheme works:

$$\begin{aligned} x(t)\cos(2\pi f_c t) &= m_1(t)\cos^2(\omega_c t) + m_2(t)\sin(\omega_c t)\cos(\omega_c t) \\ &= \frac{1}{2}[m_1(t) + m_1(t)\cos(2\omega_c t)] + \frac{1}{2}m_2(t)\sin(2\omega_c t) \\ &\rightarrow \frac{1}{2}m_1(t) \text{ after passing through LPF,} \end{aligned}$$

and similarly,

$$\begin{aligned} x(t)\sin(\omega_c t) &= m_1(t)\cos(\omega_c t)\sin(\omega_c t) + m_2(t)\sin^2(\omega_c t) \\ &= \frac{1}{2}m_1(t)\sin(2\omega_c t) + \frac{1}{2}[m_2(t) - m_2(t)\cos(2\omega_c t)] \\ &\rightarrow \frac{1}{2}m_2(t) \text{ after passing through LPF.} \end{aligned}$$