Problem 1 *(Short questions.)* 20 Points

For each of the following statements, if you believe it is true, give a justification. If you believe it is false, give a counterexample.

(a) A linear causal continuous-time system is always time-invariant.

$y(t) = t\,x(t)$ is linear, causal, and continuous-time but it is definitely not time-invariant.

$\boxed{\text{False.}}$

(b) The system with (real-valued) input $x(t)$ and output given by

$$y(t) = (1 + x^2(t))^{\cos(t)} \tag{1}$$

is stable.

$\boxed{\text{True.}}$ $|x(t)| < M$ where $M \in \mathbb{R}_+$. Thus, $\left(1 + x^2(t)\right) < M^2 + 1$. Also, note $1 \leq (1 + x^2(t)) < M^2 + 1$. Since $-1 \leq \cos(t) \leq 1$, it follows that $\frac{1}{M^2+1} < \left(1 + x^2(t)\right)^{\cos(t)} < M^2 + 1$. Thus, $y(t)$ is bounded and the system is stable. The key point here is that the base, $(1 + x^2(t))$, is never less than 1. If the base was allowed to approach 0, as $\cos(t)$ went to -1, $y(t)$ would go to ∞.

(c) The discrete-time signal $x[n] = \cos(n)$ is a periodic signal.

$\boxed{\text{False.}}$ For the sake of a contradiction, let us assume $\cos(n)$ is periodic. The first period would end the first time $n = 2\pi m$ for $m \in \mathbb{Z}_+$. This implies $\pi = \frac{n}{2m}$ for some $m, n \in \mathbb{Z}_+$. If this were possible, π would be a rational number which is certainly not true.

(d) For an otherwise completely unknown system, it is known that when the input is given by

$$x(t) = \cos(t) + \cos(2t), \tag{2}$$

the output is

$$y(t) = \frac{1}{2}\left(1 + \cos(t) + \cos(2t) + \cos(3t)\right). \tag{3}$$

This system cannot be a linear time-invariant (LTI) system.

$\boxed{\text{True.}}$ $x(t) = \cos(t) + \cos(2t) = \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} + \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}$

Recall that complex exponentials are eigenfunctions for LTI systems. Thus, the output should be composed of the original input complex exponentials scaled by their eigenvalues.

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI SYSTEM}} \longrightarrow \lambda e^{j\omega t} \qquad \lambda \in \mathbb{C}$$

$$\lambda = H(j\omega)$$

However, $y(t) = \frac{1}{2}\left(1 + \cos(t) + \cos(2t) + \cos(3t)\right)$

$$y(t) = \frac{1}{2}\left(1 + \frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt} + \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t} + \frac{1}{2}e^{j3t} + \frac{1}{2}e^{-j3t}\right)$$

$$y(t) = \frac{1}{2} + \underbrace{\frac{1}{4}e^{jt} + \frac{1}{4}e^{-jt} + \frac{1}{4}e^{j2t} + \frac{1}{4}e^{-j2t}}_{} + \frac{1}{4}e^{j3t} + \frac{1}{4}e^{-j3t}$$

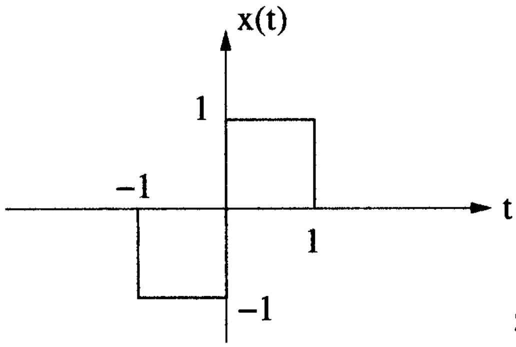
Only these terms could be generated by scaling our input complex exponentials, thus the system is not LTI.

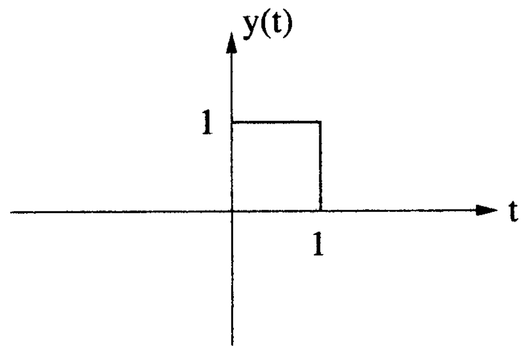
Problem 2 *(Convolution.)* 20 Points

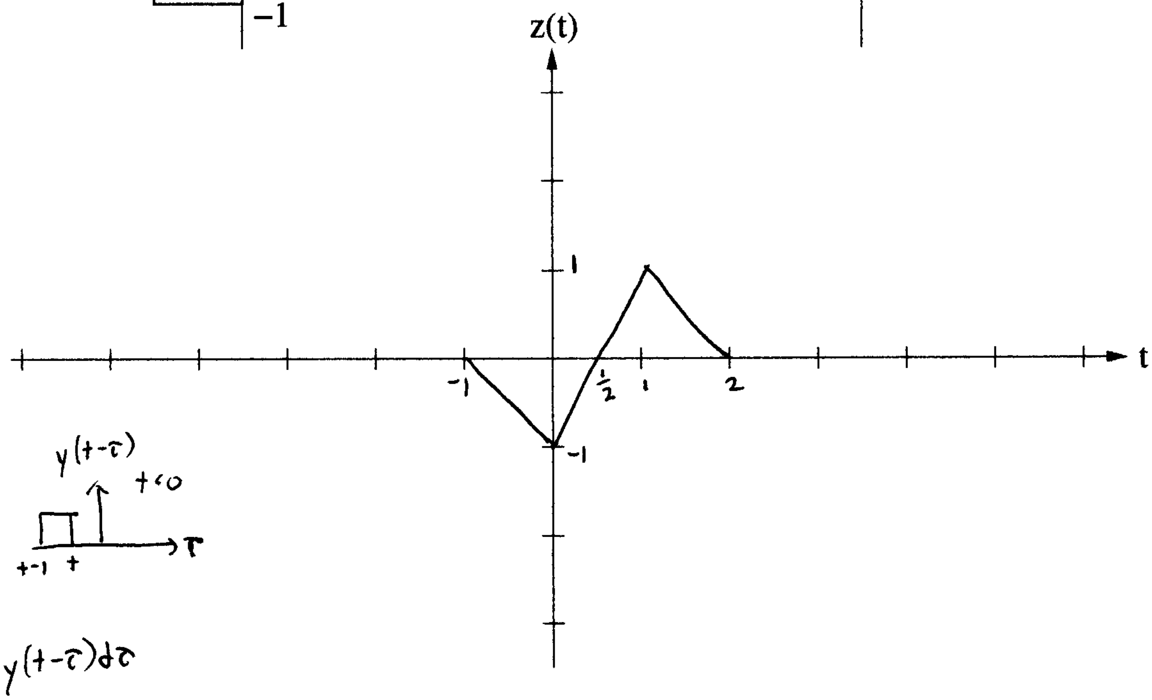
The continuous-time signals $x(t)$ and $y(t)$ are given in Figure 1. In the figure, draw the signal $z(t)$ given by

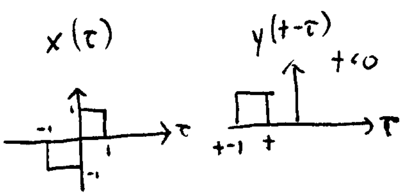
$$z(t) \;=\; (x * y)(t). \tag{4}$$

Carefully label both axes.









$z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

Figure 1: Convolution: $z(t) = (x * y)(t)$.

$\underline{t < -1}$

$z(t) = 0$

$\underline{-1 \le t < 0}$

$z(t) = \int_{-1}^{t} -1 \, d\tau = -\tau \Big|_{-1}^{t} = -t-1$

$\underline{0 \le t < 1}$

$z(t) = \int_{t-1}^{0} -1 \, d\tau + \int_{0}^{t} 1 \, d\tau = t-1+t = 2t-1$

$\underline{1 \le t < 2}$

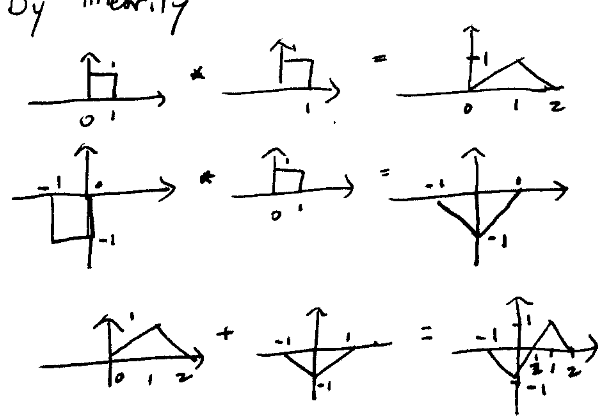
$z(t) = \int_{t-1}^{1} 1 \, d\tau = 1-t+1 = 2-t$

$\underline{t \ge 2}$

$z(t) = 0$

$\boxed{\text{OR}}$

By linearity



$$z(t) = \begin{cases} 4 & \\ -t-1 & -1 \le t < 0 \\ 2t-1 & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & \text{o/w} \end{cases}$$

Problem 3 *(Inverse discrete-time Fourier transform.)* 15 Points

A discrete-time signal $h[n]$ has discrete-time Fourier transform

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}. \tag{5}$$

Find the signal $h[n]$.

Recall that $\dfrac{1}{1-ae^{-j\omega}} \xrightarrow{F^{-1}} a^n u[n]$ if $|a|<1$.

$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$$

Also, recall that $X(e^{j\omega})e^{-j\omega n_0} \xrightarrow{F^{-1}} x[n-n_0]$ $\forall n_0 \in \mathbb{Z}$.

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= \left(\frac{1}{2}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n-1]$$

$$= \delta[n] + 3\left(\frac{1}{2}\right)^n u[n-1]$$

These are all equivalent.

Problem 4 *(A linear time-invariant system.)* 30 Points

A linear time-invariant system with input $x(t)$ and output $y(t)$ satisfies

$$\underbrace{a^2 y(t) + 2a\frac{dy(t)}{dt} + \frac{d^2y(t)}{dt^2} \quad = \quad x(t).}_{} \tag{6}$$

(a) (10 Points) Find the frequency response $H(j\omega)$ of the considered system.

$$a^2 Y(j\omega) + 2a\, j\omega\, Y(j\omega) + (j\omega)^2 Y(j\omega) = X(j\omega)$$

REWRITE: $$Y(j\omega)\left(a^2 + 2aj\omega + (j\omega)^2\right) = X(j\omega)$$

$$Y(j\omega)\,(a+j\omega)^2 = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(a+j\omega)^2}$$

ALSO EQUAL TO $$= \frac{1}{a^2 + 2aj\omega - \omega^2}$$

(b) (10 Points) For $a = 1/2$, sketch the magnitude of the frequency response $H(j\omega)$. Is the system rather high-pass or rather low-pass? Justify your answer.

$$|H(j\omega)| = \frac{1}{|(a+j\omega)|^2} = \frac{1}{|(a+j\omega)(a+j\omega)|}$$

$$= \frac{1}{|a+j\omega|\ |a+j\omega|} = \frac{1}{\sqrt{a^2+\omega^2}\ \sqrt{a^2+\omega^2}}$$

$$= \frac{1}{a^2+\omega^2}$$

FOR $a = 1/2$:

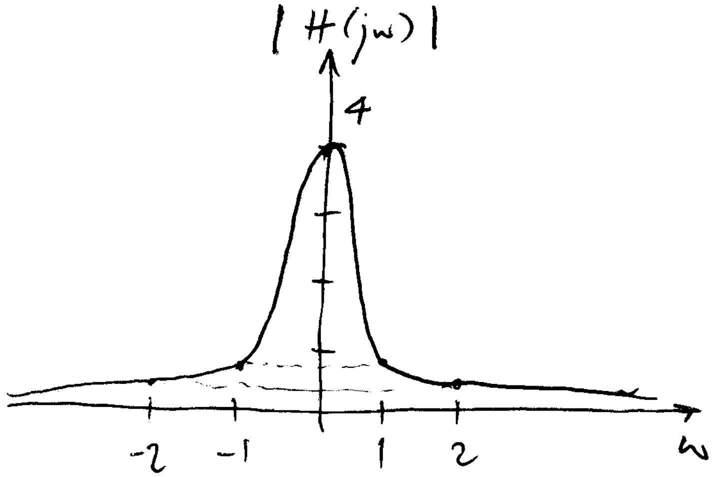
$$|H(j\omega)| = \frac{1}{1/4 + \omega^2}$$

$\omega = 0 \rightarrow |H(j0)| = 4$

$|\omega| = 1 \Rightarrow |H(j1)| = 4/5$

$|\omega| = 2 \Rightarrow |H(j2)| = 4/9$

$|\omega| \to \infty \Rightarrow |H(j\omega)| \longrightarrow 0.$



(c) (10 Points) For what values of a is the system stable? Justify your answer. *Remark.* If you cannot solve the math, don't worry. Just describe *clearly and concisely* how you would proceed, and you will get partial credit.

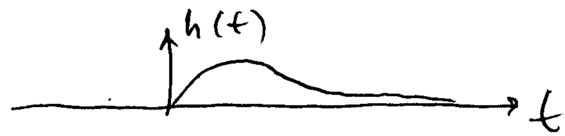
THE SYSTEM IS STABLE IF AND ONLY IF

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty .$$

$\longrightarrow$ WE NEED TO DETERMINE $h(t)$.

$\boxed{\text{CASE 1: } a > 0}$

FROM TABLE: $\dfrac{1}{(a+j\omega)^2} \bullet\!\!-\!\!\!-\!\!\circ\; te^{-at}u(t)$



$$\int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} |te^{-at}u(t)|\, dt = \int_0^{\infty} |t e^{-at}|\, dt$$

$$= \int_0^{\infty} t e^{-at}\, dt \qquad \text{SINCE INTEGRAND IS ALWAYS NON-NEGATIVE}$$

$$= \left[\frac{t}{-a}e^{-at} - \frac{1}{a^2}e^{-at}\right]_0^{\infty} \qquad \text{INTEGRATION BY PARTS}$$

$$= 1/a^2 \Rightarrow \boxed{\text{SYSTEM IS STABLE}}$$

$\boxed{\text{CASE 2: } a = 0}$: $H(j\omega) = \dfrac{1}{(j\omega)^2}$

FROM TABLE: $u(t) - 1/2 \;\circ\!\!-\!\!\!-\!\!\bullet\; \dfrac{1}{j\omega}$

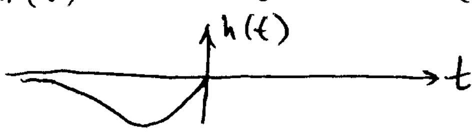
DIFF. IN FREQ: $t(u(t) - 1/2) \;\circ\!\!-\!\!\!-\!\!\bullet\; \dfrac{1}{(j\omega)^2}$

$$\int_{-\infty}^{\infty} |t\,(u(t) - 1/2)|\, dt = \infty \Rightarrow \boxed{\text{SYSTEM IS UNSTABLE}}$$

$\boxed{\text{CASE 3: } a < 0}$ PROBLEM: te^{-at} BLOWS UP FOR $t > 0$.

SO, TRY $t < 0$ INSTEAD!

$$h(t) = -te^{-at}u(-t) \;\circ\!\!-\!\!\!-\!\!\bullet\; H(j\omega) = \frac{1}{(a+j\omega)^2}$$



SAME AS $\boxed{\text{CASE 1}}$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{0} te^{-at}\, dt \stackrel{\downarrow}{=} 1/a^2$$

$$\Rightarrow \boxed{\text{SYSTEM IS STABLE}}$$

Problem 5 *(Filtering.)* 15 Points

The signal $x(t)$ with spectrum $X(j\omega)$ as shown in Figure 2 is passed through a linear time-invariant (LTI) system with impulse response

$$h(t) = 2\mathrm{sinc}(2t), \tag{7}$$

where, as defined in class,

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}. \tag{8}$$

Denote the output of the system by $y(t)$. Calculate the error between $x(t)$ and $y(t)$, given by

$$\int_{-\infty}^{\infty} |x(t) - y(t)|^2\, dt. \tag{9}$$

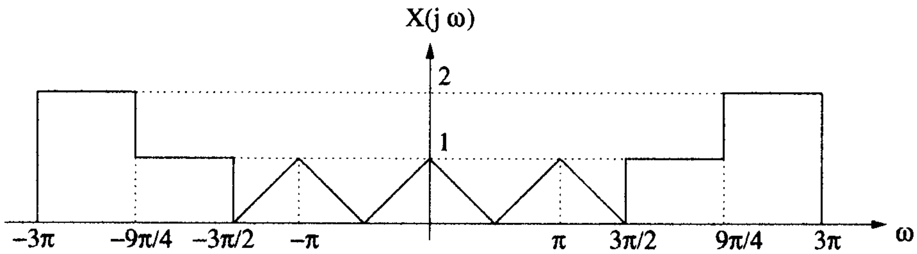
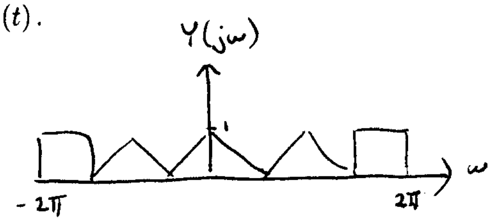


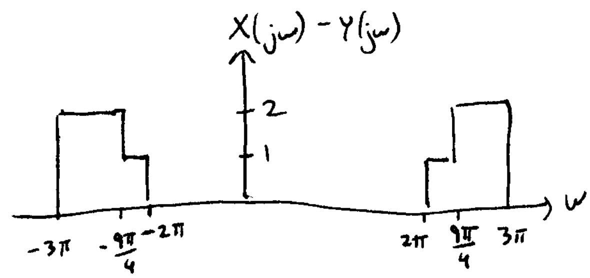
Figure 2: The spectrum of the signal $x(t)$.

Recall that $\dfrac{\sin(Wt)}{\pi t} \xrightarrow{\text{CTFT}} \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$.



$h(t) = 2\,\mathrm{sinc}(2t) = \dfrac{\cancel{2}\sin(\pi 2t)}{\pi \cancel{2}t} = \dfrac{\sin(2\pi t)}{\pi t}$

$H(j\omega) = \begin{cases} 1 & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases}$



$$\int_{-\infty}^{\infty} |x(t)-y(t)|^2\,dt \overset{\text{Parseval's}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)-Y(j\omega)|^2\,d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-3\pi}^{-\frac{9\pi}{4}} 2^2\,d\omega + \int_{-\frac{9\pi}{4}}^{-2\pi} 1^2\,d\omega + \int_{2\pi}^{\frac{9\pi}{4}} 1^2\,d\omega + \int_{\frac{9\pi}{4}}^{3\pi} 2^2\,d\omega\right) = \frac{1}{2\pi}\left(2\cdot 4\cdot \frac{3\pi}{4} + 2\cdot 1\cdot \frac{\pi}{4}\right)$$

$$= 3 + \frac{1}{4} = \boxed{\frac{13}{4}}$$