Problem 1 (45 pts)

[10 pts]
a) \(y[n] = 2x[n] - 0.5y[n-1]\)
\(y[n] + 0.5y[n-1] = 2x[n]\)

[15 pts]
b) Easy way: use \(h[n]\) from part (c):
\[x[n] = \sum_{k=-\infty}^{n} h[k] = \sum_{k=-\infty}^{n} 2(-0.5)^k u[k] = \sum_{k=0}^{n} (-0.5)^k\]
For \(n < 0\), \(s[n] = 0\)
For \(n \geq 0\): use \(\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}\)
\(s[n] = 2 \times \frac{(1-(-0.5) \times (n+1))/(1+(-0.5)) = 2 \times (2/3 + 1/3 \times (-1/2)^n)}\)
For all \(n\): \(s[n] = (4/3 + 2/3(-1/2)^n) u[n]\)

Hard way: solve difference equation
\(y[n] + 0.5y[n-1] = 2x[n]\)
\(x[n] = u[n]\), zero-initial conditions: \(y[-1] = 0\)
Homo. soln: \((y^n)[n] = 0.5(y^n)[n-1] = 0\)
Char eqn: \(r + 0.5 = 0\)
\((y^n)[n] = A (-0.5)^n, n \geq 0\)
Part. soln: \(x[n] = 1, n \geq 0\)
\((y^p)[n] = c, n \geq 0\)
Total soln: \(y[n] = 4/3 + A(-0.5)^n, n \geq 0\)
Translate initial condition:
\(y[n] = -0.5y[n-1] + 2x[n]\)
For \(n = 0\):
\(y[0] = -0.5y[-1] + 2x[0]\)
\(y[0] = -0.5 \times 0 + 2 = 2\)
Find \(A\) by satisfying initial condition:
\(y[0] = 2 = 4/3 + A \times (-0.5)^0 = 2\)
\(A = 2/3\)
\(y[n] = s[n] = 4/3 + 2/3 \times (-0.5)^n, n \geq 0\)
Since \(s[n] = 0, n < 0,\)
\(s[n] = (4/3 + 2/3(-0.5)^n) u[n]\)

[10 pts]
c) Find a closed-from expression for the impulse response \(h[n]\) for all \(n\). (You can do this using the result of part (b).
Alternatively you can write down the result by inspection.

Easy way (inspection of block diagram):
\(y[n] = h[n]\) when \(x[n] = \delta[u]\) and \(y[-1] = 0\)
Under these conditions:
\[ y[0] = 2 \quad \text{due to input} \]
\[ y[1] = 2(-.5) \quad \text{due to fed-back output} \]
\[ y[2] = 2(-.5)(-.5) \quad \text{due to fed-back output} \]

\[ y[n] = h[u] = 2 * (-.5)^n, \quad n \geq 0 \]

since \( h[n] = 0, \quad n < 0 \), \( h[n] = 2(-.5)^n \cdot u[n] \)

Hard way from \( s[n] \):
\[ h[n] = s[n] - s[n-1] \]
\[ = \left[ \frac{4}{3} + \frac{2}{3}(-.5)^n \right] y[n] - \left[ \frac{4}{3} + \frac{2}{3} (-.5) ^{(n-1)} \right] y[n-1] \]

\( n = 0: \quad h[0] = 2 \)
\( n \geq 1: \quad h[n] = \frac{4}{3} + \frac{2}{3}(-.5)^n - \frac{4}{3} - \frac{2}{3} (-.5) ^{(n-1)} \)
\[ h[n] = 2 (-.5)^n \cdot u[n] \]

[10 pts]

let \( w = \omega \)

d) \( y[n] + .5y[n-1] = 2x[n] \)
\[ x[n] = e^{jwn}, \quad y[n] = H(e^{jw})e^{jwn} \]
\[ H(e^{jw})e^{jwn} + .5 H(e^{jw})e^{jwn}e^{-jw} = 2e^{jwn} \]
\[ H(e^{jw}) = \frac{2}{1 + .5e^{-jw}} \]

Problem 2 (35 pts)

[10 pts]

a)

[15 pts]

let \( w = \omega_0 \)

b) \( N = 4, \quad w = \pi/2 \)
\[ x[n] = \frac{1}{N} \sum_{n=-2}^{1} \{ x[n]e^{-jwn} \} \]
\[ = \frac{1}{4} \sum_{n=-2}^{1} \{ x[n]e^{-jwn} \} \]
\[ = \frac{1}{4} \left[ -e^{jk*\pi/2} + e^{-jk*\pi/2} \right] \]
\[ = -j/2 \cdot \sin(k*\pi/2) \]

[10 pts]

c) \( \text{abs}(X[k]) = .5 \text{abs}(\sin(k*\pi/2)) \)
\[ \arg\{X[k]\} = \arg\{-j/2\} + \arg\{\sin(k*\pi/2)\} \]
\[ = -\pi/2 \quad \text{if} \ \sin(k*\pi/2) > 0 \]
\[ \pi \quad \text{if} \ \sin(k*\pi/2) < 0 \]

When \( \text{abs}(X[k]) = 0 \), it doesn't matter what you choose for \( \arg \).
Problem 3

\[ h(t) = u(t-1) - u(t-2) \]
\[ x(t) = (e^t) u(t) \]
\[ y(t) = x(t) * h(t) = (e^t) u(t) \]
\[ = [(e^t) u(t)] * [\delta(t-1) - \delta(t-2)] \]
\[ (e^t) u(t) = \int_{-\infty}^{t} (e^\tau u(\tau) \, d\tau) \]
\[ = (e^t) - 1 \quad \text{if } t \geq 0 \]
\[ = 0 \quad \text{if } t < 0 \]
\[ = [(e^t) - 1] u(t) \]
\[ y(t) = [e^{(t-1)} - 1] u(t-1) - [e^{(t-2)} - 1] u(t-2) \]