Problem 1 (15 pts.) The equivalent noise bandwidth of a CT LTI system $h(t) \leftrightarrow H(j\omega)$ is defined as:

$$BW = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|H(j\omega)|^2}{|H(j0)|^2} d\omega.$$ 

Find the value of BW for the system having the following impulse response.

$h(t) = 0$ for $t < 0$ and $t > 6$
Problem 2  (25 pts.) The CT LTI system shown here can model many simple situations that produce echoes. Here, $K$ and $\Delta$ are real constants, and $\Delta \geq 0$.

(a)  (10 pts.) Find an expression for the impulse response $h(t)$.

(b)  (10 pts.) For what values of $K$ is the system stable? Justify your answer.

(c)  (5 pts.) Let $K = -1$, and assume $x(t) = u(t) - u(t - 2\Delta)$. Sketch $y(t)$.
Problem 3  (40 pts.) Consider a DT system described by the difference equation:

\[ y[n] + y[n-1] - 2y[n-2] = x[n] + x[n-1]. \]

(a)  (5 pts.) Sketch a realization of this system using only two delay elements.

(b)  (5 pts.) Find the transfer function \( H(z) \).

(c)  (5 pts.) Plot the poles and zeros and indicate the region of convergence of \( H(z) \).

(d)  (5 pts.) Is the system BIBO stable? Justify your answer.
(e) (5 pts.) Find the impulse response $h[n]$.

(f) (5 pts.) Let the input be $x[n] = 3^n$, $\forall n$. Find $y[n]$, $\forall n$.

(g) (10 pts.) Let the input be $x[n] = 3^n u[n]$, $\forall n$. (If you are interested in the initial conditions, they are fully specified by our specification of the input signal.) Find $y[n]$, $\forall n$. Explain why, as $n \to \infty$, $y[n]$ doesn’t agree with the result found in part (f).
Problem 4 (45 pts.) In the diagram below, we process a CT signal \( x(t) \) using a CT LTI system \( h_c(t) \leftrightarrow H_c(s) \) to obtain the output \( y_c(t) \). (Here, the subscript \( c \) means that \( y_c(t) \) was processed using a CT system).

Alternatively, we can sample \( x(t) \) at some appropriate rate \( 1/T \) to obtain a DT signal \( x[n] \), process this using a DT LTI system \( h_d[n] \leftrightarrow H_d(z) \) to obtain \( y_d[n] \), and perform reconstruction to obtain the CT signal \( y_d(t) \). (Here, the subscript \( d \) means that \( y_d(t) \) was processed using a DT system).

How to choose \( h_d[n] \leftrightarrow H_d(z) \) so that its effect is similar to that of \( h_c(t) \leftrightarrow H_c(s) \) is a major topic in the study of digital signal processing. Here, we consider one technique, which makes reference to the diagrams below.

We choose some “typical” input signal \( x(t) \), and then choose \( h_d[n] \leftrightarrow H_d(z) \) so that \( y_d[n] = y_c[n] \). It should be emphasized that given the resulting \( h_d[n] \leftrightarrow H_d(z) \), the condition \( y_d[n] = y_c[n] \) is satisfied only for that particular choice of \( x(t) \). In this problem, we choose \( x(t) = u(t) \), which is a common choice, and we say that \( h_d[n] \leftrightarrow H_d(z) \) has been obtained from \( h_c(t) \leftrightarrow H_c(s) \) by a “step-invariant transformation”. (We are not concerned by the fact that the unit step function is not strictly bandlimited.)
(a) (5 pts.) Consider a causal CT system with impulse response \( h_c(t) \). Show that the step-invariant \( h_d[n] \xleftrightarrow{Z} H_d(z) \) are specified by the relations:

\[
\begin{align*}
    u[n]*h_d[n] &= [u(t)*h_c(t)]|_{t=nT} \quad \text{and} \quad H_d(z) = (1-z^{-1})Z\{[u(t)*h_c(t)]|_{t=nT}\}.
\end{align*}
\]

(b) (5 pts.) Consider the causal, first-order CT system with transfer function:

\[
H_c(s) = \frac{a}{s+a}.
\]

Sketch the poles and zeros of \( H_c(s) \) in the \( s \)-plane, and state the condition on \( a \) for this system to be stable.
(c) (15 pts.) Using the results of part (a), find the step-invariant $H_d(z)$ for the $H_c(s)$ given in part (b). Sketch the poles and zeros of $H_d(z)$ on the $z$-plane as a function of $a$ and $T$, and state the conditions on $a$ and $T$ for the system described by $H_d(z)$ to be stable. You may find it useful to define $\alpha = e^{-aT}$.

(d) (5 pts.) Consider the causal, second-order CT system with transfer function:

$$H_c(s) = \frac{s}{s^2 + 2s + 101}.$$ 

Sketch the poles and zeros of $H_c(s)$ in the $s$-plane, and state whether the system is stable. What type of system is this (e.g., integrator, highpass filter, etc.)?
(e) (15 pts.) Suppose \( T = 0.02 \) s. Using the results of part (a), find the step-invariant \( H_d(z) \) for the \( H_c(s) \) given in part (d). Sketch the poles and zeros of \( H_d(z) \) on the \( z \)-plane, and state if the system described by \( H_d(z) \) is stable. You may want to define \( r = e^{-T} \approx 0.98 \) and \( \theta = 10T = 0.2 \) rad \( \approx 11^\circ \).
Problem 5 (35 pts.) Let a CT LTI system be described by $h(t) \leftrightarrow H(j\omega)$. Suppose that $H(j\omega)$ is periodic in $\omega$, i.e., $H(j(\omega + \omega_0)) = H(j\omega), \forall \omega$, for some $\omega_0$. Describe $h(t)$ as precisely as you can, and show how to express $h(t)$ in terms of an integral of $H(j\omega)$ over one period, e.g., $0 \leq \omega < \omega_0$. *Hint:* it might help to first think about some specific example(s).
Problem 6  (40 pts.) This problem concerns sampling and reconstruction, i.e., conversion of CT signals to DT and back again. To simplify the mathematics, we will consider an equivalent all-CT system. The signal \( x(t) \leftrightarrow X(j\omega) \) is bandlimited to \(|\omega| < 2\pi/T\). Samples are taken of \( x(t) \) and of \( x(t) * h(t) \) at a rate of \( 1/T \), half the rate that would be required to specify \( x(t) \) if only \( x(t) \) were sampled. This problem shows that for some \( h(t) \leftrightarrow H(j\omega) \), it is possible to find \( G_1(j\omega) \) and \( G_2(j\omega) \) so that \( y(t) = x(t) \).

(a)  (15 pts.) Find an expression for \( Y(j\omega) \).

(b)  (5 pts.) Find a simpler expression for \( Y(j\omega) \) that is valid for \(-2\pi/T \leq \omega \leq 2\pi/T\), i.e., including only those terms that contribute over that frequency range.
(c) (20 pts.) Show that if \( x(t) * h(t) = dx/dt \) and if \( G_1(j\omega) \) and \( G_2(j\omega) \) are as specified, then
\[ y(t) = x(t). \] 
Hint: it is probably easiest to prove this using a series of carefully labeled sketches. Please be honest in answering the question. If you make an error in some intermediate step but miraculously obtain the desired answer, you will receive no credit.