Key.

## Problem 1 LTI Properties (26 pts)

[24 pts] Classify the following systems, with input $x(t)$ (or $x[n]$) and output $y(t)$ (or $y[n]$). In each column, write "yes", "no", or "?" if the property is not decidable with the given information. (+1 for correct, 0 for blank, -0.5 for incorrect). (For 1d, you are given the system is known to be linear and time=invariant.) For 1b and 1d, 2 test input cases are given.

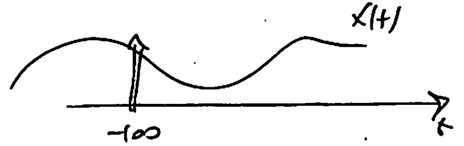
Let $\Pi(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$

| System | Causal | Linear | Time-invariant | BIBO stable |
|---|---|---|---|---|
| a. $y(t) = 2x(t-1) - 5$ | yes | no | yes | yes |
| b. $y(t) = \begin{cases} 0 & \text{if input } x(t) = 0 \\ tu(t) & \text{if input } x(t) = u(t-1) \end{cases}$ | ? | ? | ? | no |
| c. $y(t) = x(t)[\cos(2\pi t)u(t)]$ | yes | yes | no | yes |
| d. $y(t) = \begin{cases} 0 & \text{if input } x(t) = 0 \\ u(t) & \text{if input } x(t) = u(t) \end{cases}$ | yes | YES | YES | yes |
| e. $y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\Pi(t-\tau)d\tau$ | no | yes | yes | yes |
| f. $y(t) = x(t) \cdot [1 - \delta(t+100)]$ | yes | yes | no | no |
| g. $y[n] = \mathcal{Z}^{-1}\{\frac{z^2}{z+1}\} * x[n]$ | no | yes | yes | no |

e) $y(t) = x(t) * \Pi(t)$

f) $x(t) - x(t)\delta(t+100)$

$= x(t) - x(t+100)\delta(t+100)$



g) $\frac{z^2}{z+1} = \frac{z}{1+z^{-1}} \xrightarrow{\mathcal{Z}^{-1}\{\}} \delta[n+1] * (-1)^n u[n] = (-1)^{(n+1)} u[n+1]$

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Key.

## Problem 2 Short Answers ~~(29~~ 40 pts)

Answer each part independently. Note $\Pi(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$.

[4 pts] a. Complete the table with the appropriate type of Fourier transform to use (FS, FT, DTFT, or DFT) on a signal of each type.

| | aperiodic in time | periodic in time |
|---|---|---|
| continuous time | FT | FS |
| discrete time | DTFT | DFT |

[3 pts] b. $X(j\omega) = \cos(\omega/2) + 1$. Find $x(t)$.

$X(j\omega) = \frac{1}{2}(e^{j\omega/2} + e^{-j\omega/2}) + 1$

$x(t) =$ $\delta(t) + \frac{1}{2}(\delta(t+1/2) + \delta(t-1/2))$.

4 [~~3~~ pts] c. A periodic signal $x(t) = p(t) * \sum_{n=-\infty}^{\infty} \delta(t-2n)$, where $\mathcal{F}\{p(t)\} = P(j\omega) = \cos(\omega/2) + 1$. The fundamental period $\omega_o = \pi$. Find the Fourier series coefficients $a_k$.

$a_k =$ $\frac{1}{2}(1 + \cos(k\pi/2))$

$p(t) * \sum \delta(t-2n) \longleftrightarrow P(j\omega) \cdot \frac{2\pi}{2} \sum \delta(\omega - \frac{2\pi k}{2}) = \underbrace{\pi P(jk\pi)}\cdot \sum \delta(\omega - \pi k)$

note $x(t) = \sum a_k e^{jk\omega_0 t} \longrightarrow \sum a_k 2\pi \delta(\omega - k\omega_0)$

$2\pi a_k = \pi P(jk\pi)$

$a_k = \frac{1}{2} P(jk\pi)$

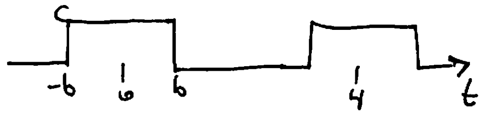
7 [~~3~~ pts] d. A periodic signal $x(t)$ has period 4 seconds and Fourier Series coefficients $a_k = \frac{\sin(k\pi/4)}{k\pi}$. Find the time average power $\frac{1}{T}\int_T x^2(t)dt$.

$\Longrightarrow \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

time average power = $1/4$

$\frac{1}{T}\int_T x^2(t)\,dt$

$= \frac{1}{4} c^2 \cdot 2b = 1/4$



$a_k = \frac{c}{4}\int_{-b}^{b} e^{-j\frac{k\pi}{2}t} dt = \frac{c}{4} \frac{2\sin\frac{k\pi}{2}\cdot b}{k\pi/2} = \frac{\sin\frac{k\pi}{4}}{k\pi} \Longrightarrow \begin{matrix} b = 1/2 \\ c = 1 \end{matrix}$

Key

[~~8~~ 9 pts] c. Initial and final value.

i. Given $X(s) = \frac{s+3}{s^2+3s+2}$. Find $x(0^+) =$ 1

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{s(s+3)}{s^2+3s+2}$$

ii. Given causal $X(z) = \frac{z^{-2}+2z^{-3}}{1-2z^{-1}+\frac{5}{4}z^{-2}-\frac{1}{4}z^{-3}}$.

Find $\lim_{n\to\infty} x[n] =$ 12

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1} (1-z^{-1})X(z)$$

$$\begin{array}{r} 1-z^{-1}+\frac{1}{4}z^{-2} \\ 1-z^{-1}\overline{\big)\,1-2z^{-1}+\frac{5}{4}z^{-2}-\frac{1}{4}z^{-3}} \\ \underline{1-z^{-1}} \\ -z^{-1}+5/4z^{-2} \\ \underline{-z^{-1}+z^{-2}} \\ \frac{1}{4}z^{-2}-\frac{1}{4}z^{-3} \end{array}$$

$$= \lim_{z\to 1} \frac{z^{-2}+2z^{-3}}{1-z^{-1}+\frac{1}{4}z^{-2}} = \frac{1+2}{1-1+1/4} = \frac{3}{1/4} = 12$$

iii. Given causal $X(z) = \frac{2+3z^{-1}}{1-2z^{-1}+\frac{5}{4}z^{-2}-\frac{1}{4}z^{-3}}$. $= \frac{2z^3+3z^2}{z^3-2z^2+\frac{5}{4}z-1/4}$

Find $x[0] =$ 2

$$\lim_{|z|\to\infty} X(z) = x[0]$$

---

[~~4~~ 5 pts] f. Given $X(s) = \frac{s+5}{s^3+3s^2+2s}$. Find $x(t) =$ $(4e^{-t}-3e^{-2t})u(t)$.

$\frac{5}{2}u(t) - 4e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t)$

$$X(s) = \frac{-3}{(s+2)} + \frac{4}{(s+1)}$$

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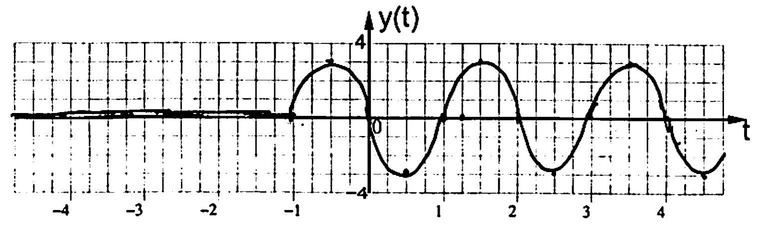
[8 pts] g. Sketch $y(t) = 3\pi \cdot u(t+1) * \cos(\pi t)u(t)$



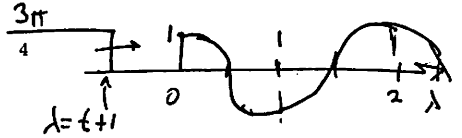
$$\int_0^{\gamma} \cos \pi t \, dt = \frac{\sin \pi t}{\pi}\Big|_0^{\gamma}$$

$$y(t) = 3\pi \int_{\lambda=-\infty}^{\infty} u(t+1-\lambda)\cos(\lambda\pi)u(\lambda)\,d\lambda = \int_{\lambda=0}^{t+1} \cos(\pi\lambda)\,d\lambda \text{ for } t>-1$$

0 else.

$t+1-\lambda>0$

$t>\lambda-1$



Key

**Problem 3. Digital Filter (~~22~~ 12 pts)**

A continuous time filter has impulse response $h(t) = e^{-\pi t/2}u(t)$.

[5 pts] a. The filter is sampled such that $h[n] = h(nT_s)$ where the sampling rate $T_s = 1$ sec. Find the Z transform of $h[n]$.

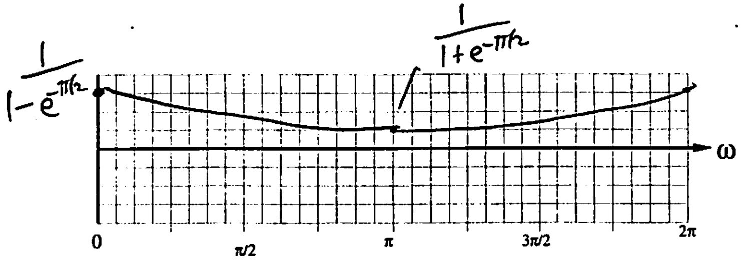
$H(z) = \frac{z}{z-e^{-\pi/2}}$

$|z| > |e^{-\pi/2}|$

$$h[n] = e^{-\pi n/2}u[n]$$

$$H(z) = \sum_{n=0}^{\infty} \left(e^{-\pi/2}z^{-1}\right)^n = \frac{1}{1-e^{-\pi/2}z^{-1}}$$

[5 pts] b. Sketch $|H(e^{j\omega})|$, labelling maximum and minimum amplitude. (Maximum and minimum may be left as functions of $e$).

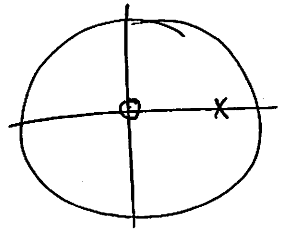


| $\omega$ | $\lvert H(e^{j\omega})\rvert$ |
|---|---|
| 0 | $\frac{1}{1-e^{-\pi/2}}$ |
| $\pi$ | $\frac{1}{\lvert -1-e^{-\pi/2}\rvert}$ |

$e^{-2} \approx .14$

$e^{-\pi/2} \approx e^{-1.5}$

$e^{-1} \approx .37$

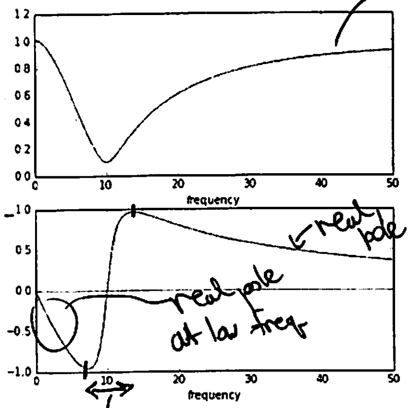


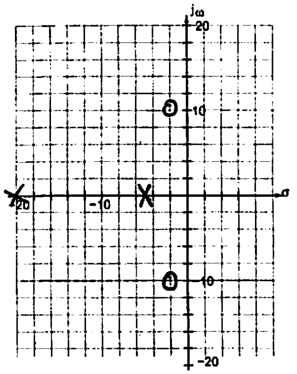
$$\frac{1}{1-e^{-\pi/2}} \approx \frac{1}{1-.25} \approx 4/3$$

$$\frac{1}{1+.25} \approx 4/5$$

## blem 4. CT and Digital Filters (22 pts)

pts] a. The magnitude and phase response for a continuous time, real, causal, stable LTI system is elow. Sketch a pole-zero diagram for a stable system (using minimum number of poles and zeros) ould match the given magnitude and phase response. equal #poles & zeros

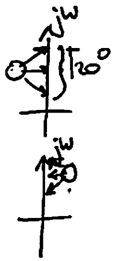




Given magnitude and phase.

Sketch corresponding pole-zero plot here.

+120° phase from -8 to 12 ⟹

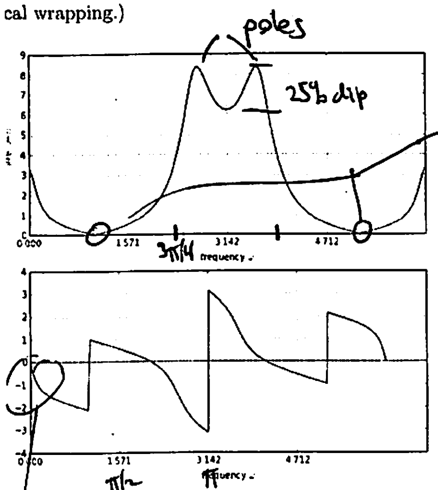


System is min phase.

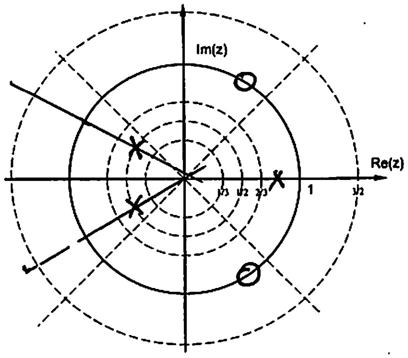
zeros ~2±10j

phase ~45° at $\omega=20$ ⟹ pole at $s=-20$

pts] b. The magnitude and phase response for a discrete time, real, causal, stable LTI system n below. Sketch a pole-zero diagram for a stable system (using minimum number of poles and which would match the given magnitude and phase response. (Note: the phase change at $\pi$ is just cal wrapping.)



zeros at $\pm\pi/3$, on unit circle



Given magnitude and phase.

Sketch corresponding pole-zero plot here.

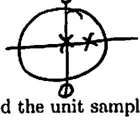
poles at $\pm\frac{5\pi}{6}$, between 1/2 and 2/3 to account for dip.

pole to account for -90° phase at small ω.

## Problem 5. Z transform (36 pts)

Key

Consider a causal DT system with



$$H(z) = \frac{z^2 + 9/4}{z(z-\frac{1}{2})}$$

$$z^2 - \tfrac{1}{2}z \overline{)\, z^2 + 9/4} \quad 1$$

$z^2 - \frac{1}{2}z$

$+\frac{1}{2}z + 9/4$

[4 pts]. a. Find the unit sample response $h[n]$ for $H(z)$.

$$H(z) = 1 + \frac{\frac{1}{2}z + 9/4}{z(z-\frac{1}{2})} = 1 + \frac{-9/2}{z} + \underbrace{\frac{5}{z-\frac{1}{2}}}$$

$h[n] =$___________

$$\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

sol #1: $\delta[n] - \frac{9}{2}\delta[n-1] + 5(\frac{1}{2})^{n-1}u[n-1]$

sol #2: $\delta[n] + (\frac{1}{2})^n u[n-1] + 9(\frac{1}{2})^n u[n-2]$

sol #3: $\delta[n] + \frac{1}{2}\delta[n-1] + 10(\frac{1}{2})^n u[n-2]$

sol #4 $(\frac{1}{2})^n u[n] + 9/4(\frac{1}{2})^{n-2}u[n-2]$

sol #5: $-9\delta[n] - \frac{9}{2}\delta[n-1] + 10(\frac{1}{2})^n u[n]$

[4 pts] b. With input $x[n]$ and output $y[n]$, find the linear difference equation in terms of $y[n]$ for this system:

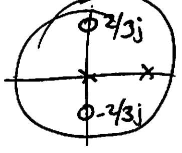
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{9}{4}z^{-2}}{1-\frac{1}{2}z^{-1}}, \quad (1-\tfrac{1}{2}z^{-1})Y(z) = (1+\tfrac{9}{4}z^{-2})X(z)$$

$y[n] = \frac{1}{2}y[n-1] + x[n] + \frac{9}{4}x[n-2]$

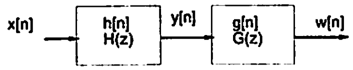
[4 pts] c. $H(z)$ is not minimum phase. Find a minimum phase function $F(z)$ such that $|H(e^{j\omega})| = |F(e^{j\omega})|$ for all $\omega$.

$$H(z) = \frac{(z+\frac{3}{2}j)(z-\frac{3}{2}j)}{z(z-\frac{1}{2})}$$

$$F(z) = \frac{(z+\frac{2}{3}j)(z-\frac{2}{3}j)}{z(z-\frac{1}{2})}$$



$$F(z) = \frac{z^2+4/9}{z(z-\frac{1}{2})} \cdot \frac{9}{4}$$



[4 pts] d. Find a stable $G(z)$ such that $|H(e^{j\omega})G(e^{j\omega})| = 1$ for all $\omega$.

$$G(z) = \frac{z(z-\frac{1}{2})}{z^2+4/9} \cdot \frac{4}{9}$$

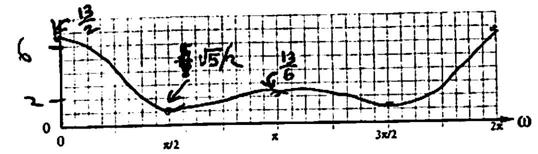
check $H(z)G(z) = \frac{z^2+9/4}{z\cdot(z-\frac{1}{2})} \cdot \frac{z(z-\frac{1}{2})}{z^2+4/9} \cdot \frac{4}{9}$

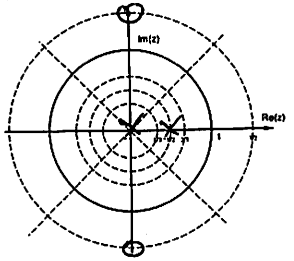
$= \frac{z^2+9/4}{z^2+4/9}\frac{4}{9} = APF$

V2

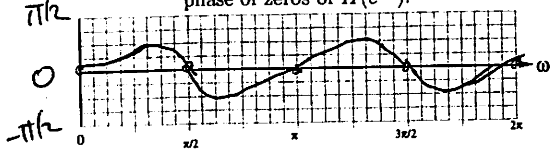
[12 pts] e. VERSION 2 Approximately sketch $|H(e^{j\omega})|$ [4 pts] and phase of the zeros *only* of $H(e^{j\omega})$ [8 pts] on the plots below, noting key maxima and minima.

$|H(e^{j\omega})|$: $\frac{|e^{2j\omega}+9/4|}{1\cdot|e^{j\omega}-1/2|}$





phase of zeros of $H(e^{j\omega})$:

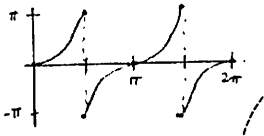


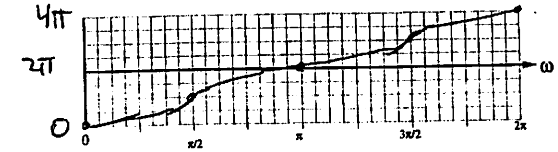
| ω | \|H\| |
|---|---|
| 0 | $\frac{13/4}{1/2} = \frac{13}{2} \approx 6.5$ |
| $\frac{\pi}{2}$ | $\frac{5/4}{\sqrt{5}/4} = \frac{\sqrt{5}}{2} \approx 1.2$ |
| π | $\frac{13/4}{3/2} = \frac{13}{6} \approx 2\frac{1}{6}$ |

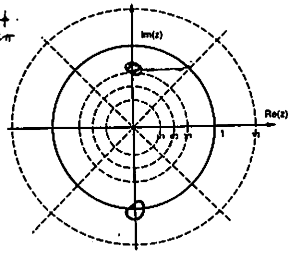
scratch work area

[8 pts] f. VERSION 2 Sketch the phase due to the zeros of $F(e^{j\omega})$ on the plot below, noting key maxima and minima. (Hint: sketch phase from each zero independently, then add.)

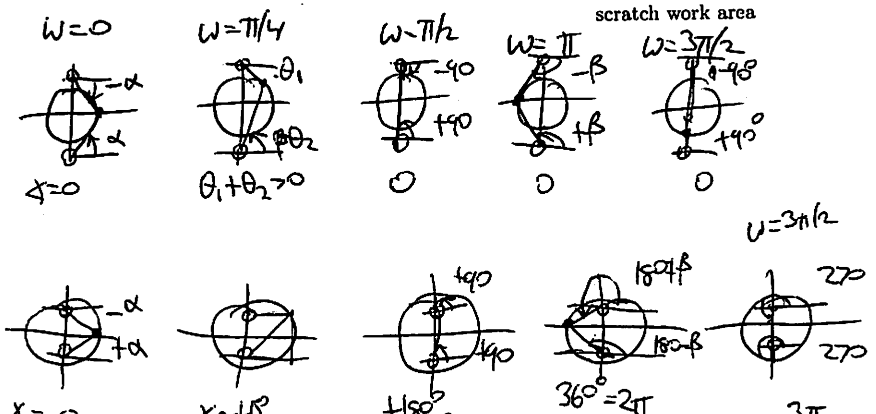
phase of zeros $\angle F(e^{j\omega})$:







scratch work area



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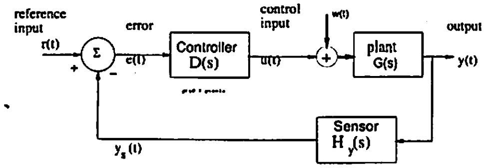
Key

## Problem 6. Control (~~24~~ 30 pts)



[2 pts] a. Find the transfer function $\frac{E(s)}{W(s)}$ in terms of $D, G, H_y$. let r(t)=0

$\frac{E(s)}{W(s)} = \frac{-GH}{1+DGH}$

$R-(DE+W)GH = E$

$E + DEGH = -WGH$

$E(1+DGH) = -WGH$

[~~8~~ 4 pts] b. Find the transfer function $\frac{Y(s)}{W(s)}$ in terms of $D, G, H_y$.

$\frac{Y(s)}{W(s)} = \frac{G}{1+GDH}$

$Y = G(W+D(R-HY))$

$Y + GDHY = GW + GDR$

$Y(1+GDH) = GW + GDR$

[10 pts] c. For the system above, let $D(s) = k_p$, $H_y(s) = 1$, and $G(s) = \frac{2\pi}{s+2\pi}$.

With input $r(t) = r_0 u(t)$, and step disturbance $w(t) = w_0 u(t)$ determine trend of $y(t)$ as $t \to \infty$.

$y(t) \to \frac{W_0 + K_p r_0}{1+K_p}$

s=0 is in ROC so FVT

$$\lim_{s\to 0} \cancel{s}\, \frac{2\pi}{s+2\pi(1+k_p)} \left[\frac{w_0}{\cancel{s}} + \frac{k_p r_0}{s}\right]$$

$$= \frac{2\pi}{2\pi(1+k_p)}(w_0 + k_p r_0)$$

$Y = \frac{G}{1+GDH} W + \frac{GD}{1+GDH} R$

$$= \frac{\frac{2\pi}{s+2\pi}}{1+\frac{2\pi}{s+2\pi}k_p} W + \frac{2\pi k_p}{s+2\pi+2\pi k_p} R$$

Key

[10 pts] d. For the system above, let $D(s) = k_p$, $H_y(s) = 1$, and $G(s) = \frac{2\pi}{s+2\pi}$.

With input $r(t) = 0$, and disturbance $w(t) = \cos(2\pi t)u(t)$, determine the sinusoidal steady state re- ponse for $y(t)$ after transients have decayed. (Hint: $y(t)$ will be of the form $M\cos(2\pi t + \phi)$. Determine 1 and $\phi$.)

$y(t) \approx$ $M\cos(2\pi t + \phi)$

$$W(t) = \frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right)u(t).$$

$$Y(s) = \frac{G}{1+GDH} W(s) = \frac{2\pi}{s+2\pi(1+k_p)} W(s)$$

$$M = \frac{1}{\sqrt{1+(1+k_p)^2}}$$

$$y(t) = \frac{2\pi}{j2\pi + 2\pi(1+k_p)} e^{j2\pi t} = \frac{1}{j+1+k_p} e^{j2\pi t}$$

$$= \cancel{M} e^{j\phi} e^{j2\pi t}$$

$$\phi = -\tan^{-1} \frac{1}{1+k_p}$$

[10 pts] e. For the system above, let $D(s) = \frac{k_p + k_d s}{s^2+4\pi^2}$, $H_y(s) = 1$, and $G(s) = \frac{2\pi}{s+2\pi}$.

With input $r(t) = 0$, and disturbance $w(t) = \cos(2\pi t)u(t) + 0.5u(t)$, determine the steady state response or $y(t)$ after transients have decayed.

$y(t) \approx$ $\frac{2\pi^2}{4\pi^2+k_p}$

$$W(s) = \frac{s}{s^2+4\pi^2} + \frac{0.5}{s}$$

$$Y = \frac{G}{1+GDH} W = \frac{\frac{2\pi}{s+2\pi}}{1 + \frac{k_p+k_d s}{s^2+4\pi^2} \cdot \frac{2\pi}{s+2\pi}} W = \frac{2\pi(s^2+4\pi^2)}{(s+2\pi)(s^2+4\pi^2) + (k_p+k_d s)2\pi} W$$

ek into 2 pieces

$$Y(s) = \frac{2\pi(s^2+4\pi^2)}{(s+2\pi)(s^2+4\pi^2)+2\pi(k_p+k_d s)} \cdot \left[\frac{s}{s^2+4\pi^2} + \frac{0.5}{s}\right]$$

↖ goes to zero

$$\lim_{s\to 0} sY(s) = \frac{0.5 \cdot 2\pi(4\pi^2)}{2\pi(4\pi^2) + 2\pi k_p} = \frac{2\pi^2}{4\pi^2+k_p}$$

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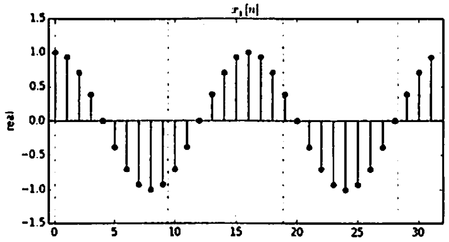
Key.

**Problem 7. DFT problem (~~28~~ 30 pts)**

[10pts] a. Given $x_1[n] = \cos(2\pi \frac{n}{16})$ as shown:



$$X_1[n] = \frac{1}{2}\left[e^{-j2\pi n/16} + e^{+j2\pi n/16}\right]$$

$$X_1[k] = \sum_{n=0}^{31} \frac{1}{2}\left(e^{+j\frac{2\pi n}{16}} + e^{-j\frac{2\pi n}{16}}\right) \cdot e^{-j2\pi nk/32}$$

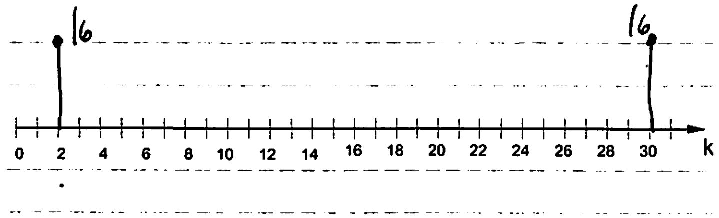
$$= \frac{1}{2} \sum_{n=0}^{31} e^{-j\frac{2\pi n}{32}(k-2)} + e^{-j\frac{2\pi n}{32}(k+2)}$$

orthogonal unless $k = \pm 2$

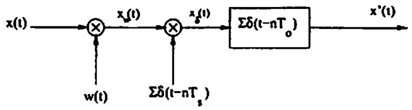
$k = -2$ or $k = 30$

sketch $X_1[k]$, the 32 point DFT of $x_1[n]$, labelling amplitudes.

$X_1[k]$:



The equivalent signal processing operations for a windowed DFT can be represented by the following block diagram:



[2pts] b. For $x_1[n]$ as given above, what are possible $T_s$ and $T_o$?

$T_s =$ 1/16 $T_o =$ 2

[2pts] c. For $X_1[k]$, what is the spacing of the frequency samples?

spacing = $\pi$ radians/second

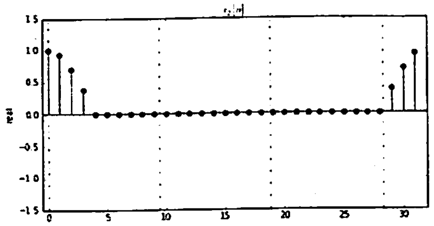
$$\frac{2\pi}{T_o} = \frac{2\pi}{2} = \pi$$

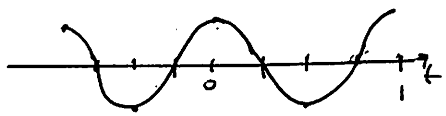
bin width $= \frac{2\pi}{N} \cdot \frac{1}{T_s} = \frac{2\pi}{T_o}$

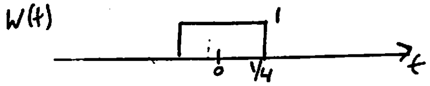
Key.

for $T_S = 1/16$ / $T_0 = 2$ sec

[16 pts] d. Given $x_2[n] = \cos(2\pi\frac{n}{16})w[n]$ as shown:

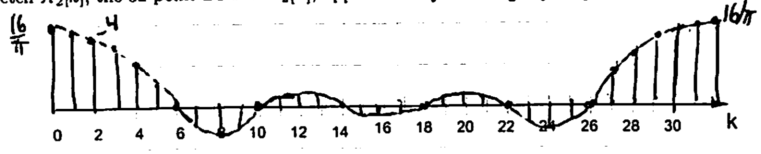






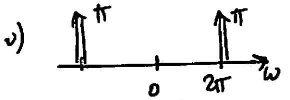
$W(j\omega) = \int_{-1/4}^{1/4} e^{-j\omega t}\,dt = \frac{2\sin\omega/4}{\omega}$

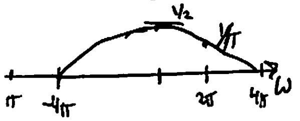
sketch $X_2[k]$, the 32 point DFT of $x_2[n]$, approximately labelling key amplitudes.



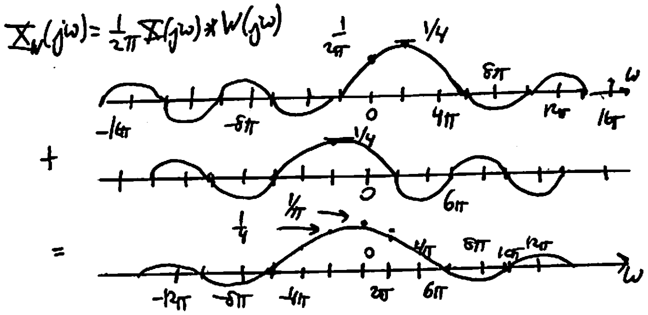
$X_2[k]$:

v)





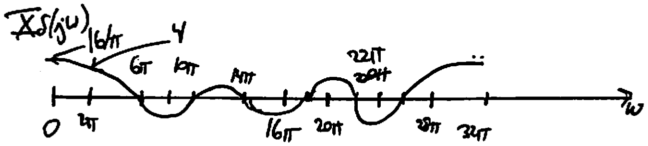
$X_w(j\omega) = \frac{1}{2\pi} X(j\omega) * W(j\omega)$

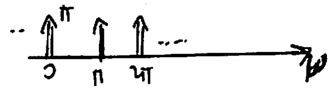


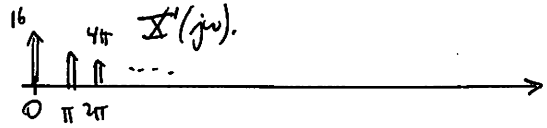
$\omega) = X_w(j\omega) * \frac{1}{2\pi}\cdot\frac{2\pi}{T_s}\sum\delta(\omega - k\frac{2\pi}{T_s})$

$\frac{2\pi}{T_s} = 32\pi$

$) = X_s(j\omega)\cdot\frac{2\pi}{T_0}\sum\delta(\omega - 32k\pi)$ $k2\pi/2$





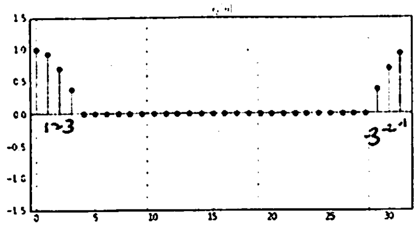


12 $X[k] = \frac{T_0}{2\pi}$ area$\{X'(j\omega)\}$

$= \frac{2}{2\pi}$ area$\{X'(j\omega)\}$.

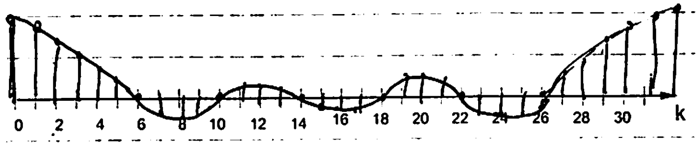
Alternate Soltn

[16 pts] d. Given $x_2[n] = \cos(2\pi\frac{n}{16})w[n]$ as shown:



sketch $X_2[k]$, the 32 point DFT of $x_2[n]$, approximately labelling key amplitudes.

$1 + 1.8 + 1.4 + .8 \approx 5$



$X_2[k]$:

$$X_2(k) = \sum_{n=-3}^{3} \cos\left(\frac{2\pi n}{16}\right) e^{-j2\pi nk/32}$$

$n = \pm 2$

$$= 1 + \cos\left(\frac{2\pi}{16}\right)\left(e^{-j2\pi k/32} + e^{+j2\pi k/32}\right) + \cos\left(\frac{2\pi\cdot 2}{16}\right)\left(e^{-j4\pi k/32} + e^{+j4\pi k/32}\right)$$

$n = \pm 1$

$$+ \cos\left(\frac{2\pi\cdot 3}{16}\right)\left(e^{-j6\pi k/32} + e^{+j6\pi k/32}\right)$$

$$= 1 + 2\cos\left(\frac{2\pi}{16}\right)\cos\left(\frac{2\pi k}{32}\right) + 2\cos\left(\frac{2\pi\cdot 2}{16}\right)\cos\left(\frac{4\pi k}{32}\right) + 2\cos\left(\frac{2\pi\cdot 3}{16}\right)\cos\left(\frac{6\pi k}{32}\right)$$

$\frac{\pi}{8}$ $\frac{\pi k}{16}$ $\frac{\pi}{4}$ $\frac{3\pi}{4}$

$\frac{3\pi k}{16}$

$\frac{k}{8}$ ∂ $3\pi/2$

16 $3\pi$

24 $\frac{9\pi}{2} = \frac{\pi}{2}$