**EE 120: Signals and Systems**
Department of EECS
UC BERKELEY

MIDTERM 3
9 December 2009

LAST Name Odic FIRST Name Pierre

Discussion Time Less discussion. More action.

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except three double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

Unabashedly, we interchange the infinite sum with the Laplace integral: $\hat{X}(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\sum_{\ell=0}^{\infty} u(t-\ell)\} = \sum_{\ell=0}^{\infty} \mathcal{L}\{u(t-\ell)\}$.

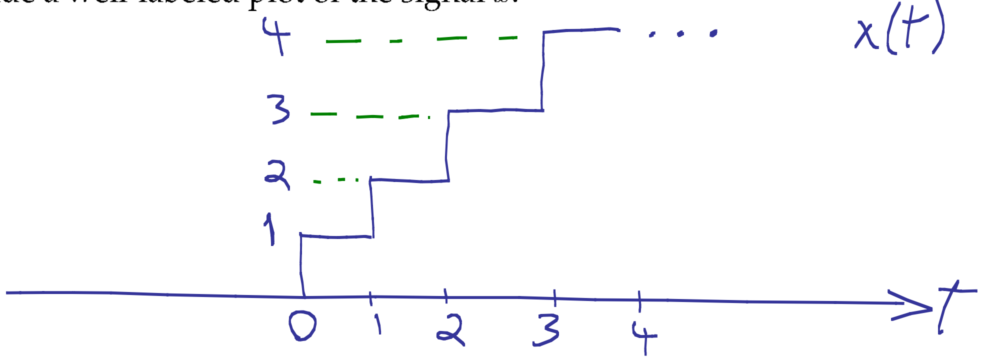
Note that the growth of $x$ is of a linear order in $t$, so we can tame it with a one-sided exponential.

**MT3.1 (40 Points)** A causal signal $x$ is defined as follows:

$$\forall t \in \mathbb{R}, \quad x(t) = \begin{cases} [t] + 1 & t \geq 0 \\ 0 & t < 0, \end{cases}$$

where $[t]$ denotes the largest integer less than or equal to $t$—that is, the value of $t$ rounded down to the largest integer. For example, $[3.01] = 3$ and $[2] = 2$.

(a) Provide a well-labeled plot of the signal $x$.




(b) Show that the Laplace transform $\widehat{X}(x)$ of the signal is given by

$$\widehat{X}(s) = \frac{1}{s(1 - e^{-s})},$$

determine its region of convergence, and provide a well-labeled plot of the pole-zero diagram for $\widehat{X}(s)$.

$$x(t) = u(t) + u(t-1) + u(t-2) + \cdots = \sum_{\ell=0}^{\infty} u(t-\ell)$$

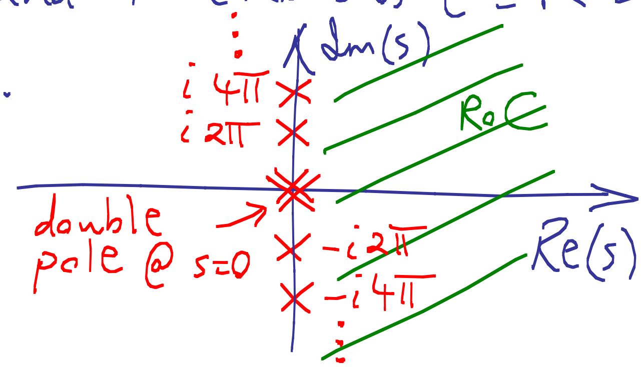
$$\hat{X}(s) = \sum_{\ell=0}^{\infty} \mathcal{L}\{u(t-\ell)\} = \sum_{\ell=0}^{\infty} \frac{e^{-\ell s}}{s} = \frac{1}{s} \sum_{\ell=0}^{\infty} (e^{-s})^{\ell} = \frac{1}{s(1-e^{-s})} \quad \text{if } |e^{-s}| < 1$$

$$e^{-s} = e^{-\sigma} e^{-i\omega} \Rightarrow |e^{-s}| = |e^{-\sigma}| < 1 \Rightarrow e^{\sigma} > 1 \Rightarrow \sigma > 0 \Rightarrow \text{RoC: } \mathrm{Re}(s) > 0$$

$$\hat{X}(s) = \frac{e^{s}}{s(e^{s}-1)} \Rightarrow$$ Poles at $s = 0$ and at the roots of $e^{s} = 1 \Leftrightarrow s = i2\pi k, \ k \in \mathbb{Z}$.

There are no zeros.

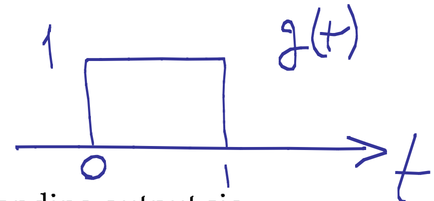
$\hat{X}(s)$ isn't rational in $s$, so don't expect #poles = #zeros



(c) We send the signal $x$ through a continuous-time LTI filter G whose impulse response $g$ is defined as follows:

$$x \longrightarrow \boxed{g} \longrightarrow y$$

$$\forall t \in \mathbb{R}, \quad g(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$




Determine, and provide a well-labeled plot of, the corresponding output signal $y$.

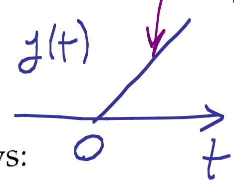
Note that $g(t) = u(t) - u(t-1)$

$$\hat{G}(s) = \frac{1}{s} - \frac{e^{-s}}{s} \Longrightarrow \hat{G}(s) = \frac{1-e^{-s}}{s} = \frac{e^{s}-1}{s\,e^{s}}$$

$$\hat{Y}(s) = \hat{X}(s)\hat{G}(s) = \frac{1}{s\cancel{(1-e^{-s})}} \frac{\cancel{1-e^{-s}}}{s}$$

$$\Rightarrow \hat{Y}(s) = \frac{1}{s^2} \longrightarrow y(t) = (u * u)(t)$$

slope = 1




(•) No pole
(•) Zeros @ $s = i2\pi k$, $k \in \mathbb{Z} - \{0\}$

Note: $e^{s} = \sum_{k=0}^{\infty} \frac{s^k}{k!} \Longrightarrow$
$e^{s} - 1 = \sum_{k=1}^{\infty} \frac{s^k}{k!}$ divisible by $s$

root of $e^{s} - 1 = 0$ at $s = 0$ cancels pole @ $s = 0$.

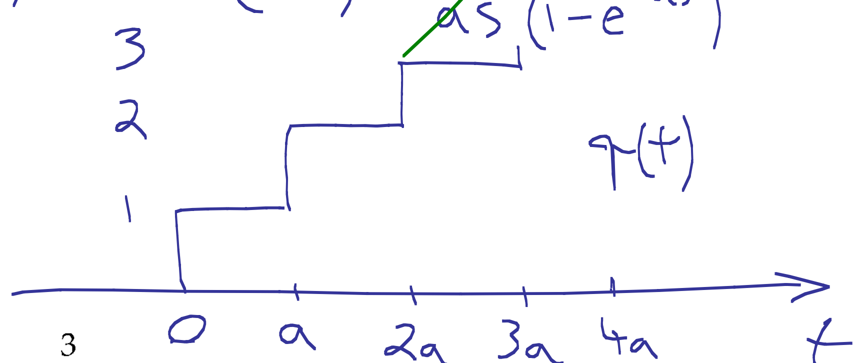
(d) A signal $q$ is defined as follows:

$$\forall t \in \mathbb{R}, \quad q(t) = \begin{cases} \left[\frac{t}{a}\right] + 1 & t \geq 0 \\ 0 & t < 0, \end{cases}$$

where $a > 0$. Determine a reasonably simple expression for its Laplace transform $\widehat{Q}(s)$. Also, determine how the plot of $q$ relates to that of $x$.

$$\left. \begin{array}{l} q(t) = x\left(\frac{t}{a}\right) \\ a > 0 \end{array} \right\} \Longrightarrow \hat{Q}(s) = a\,\hat{X}(as) = \frac{\cancel{a}}{\cancel{a}s\,(1-e^{-as})} \Longrightarrow$$

$$\boxed{\hat{Q}(s) = \frac{1}{s(1-e^{-as})}}$$



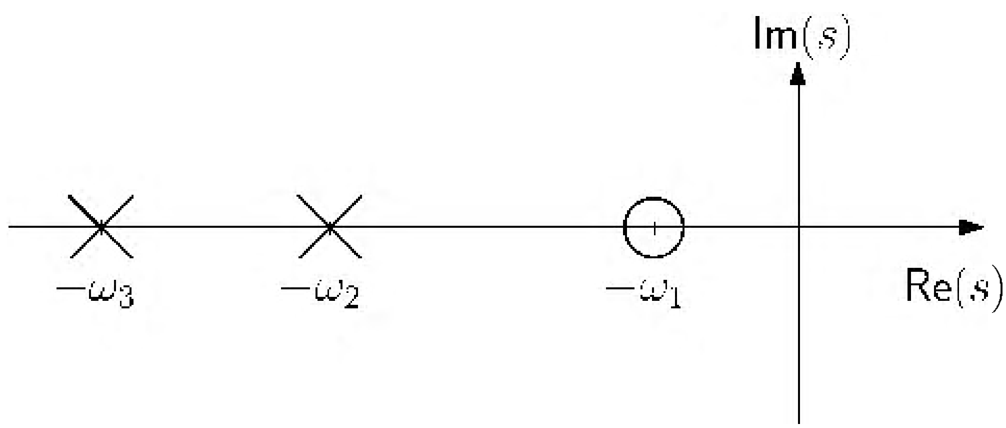



time scaling $\begin{cases} 0 < a < 1 & \text{time contraction} \\ 1 < a & \text{time dilation} \end{cases}$

**MT3.2 (40 Points)** The pole-zero diagram for a *causal* continuous-time LTI system H is shown in the figure below. We assume that $\hat{H}(s)$ is rational in $s$.

There is a single zero at $-\omega_1$, a single pole at $-\omega_2$, and a single pole at $-\omega_3$.

The frequencies $\omega_1$, $\omega_2$, and $\omega_3$ are well-separated according to the ordering $-\omega_3 < -\omega_2 < -\omega_1$. $-\omega_3 < -\omega_2 < -\omega_1$




If the input to the system is $x(t) = 1$ for all $t$, the corresponding output is $y(t) = 1$ for all $t$.

(a) Must the system be BIBO stable? Explain your reasoning.

The system is causal $\Rightarrow$ The RoC of the system function is to the right of the rightmost pole: $-\omega_2 < \mathrm{Re}(s)$.

The RoC includes the $i\omega$-axis $\Rightarrow$ The system must be BIBO stable

(b) Suppose the input to the system is the unit-step: $x(t) = u(t)$. Does the system's response have a steady-state component $y_{ss}$? If you claim that it does, explain your reasoning *and* determine the steady-state response. If you claim that it does not, explain your reasoning. Yes, the system's response will have a steady-state component, because the pole due to the unit step is at $s=0$, which is to the right of the rightmost pole of the causal system ( $u(t) \overset{\mathcal{L}}{\longleftrightarrow} 1/s$ ) $\Rightarrow$ the response due to the pole at $s=0$ dominates as $t \to \infty$. As $t \to \infty$, the system cannot distinguish between the input $x(t) = 1 \; \forall t$ and the input $x(t) = u(t) \Rightarrow y_{ss}(t) = 1 \quad t > 0$

(c) Determine a reasonably-simple expression for $\widehat{H}(s)$, the transfer function of the system.

$$\hat{H}(s) = H_0 \frac{s+\omega_1}{(s+\omega_2)(s+\omega_3)}$$

$$x(t) = 1 \; \forall t \Rightarrow y(t) = \hat{H}(s)\Big|_{s=0} 1 = \hat{H}(0) \cdot 1 = 1 \Rightarrow \hat{H}(0) = 1$$

$$\Rightarrow \frac{H_0 \, \omega_1}{\omega_2 \, \omega_3} = 1 \Rightarrow H_0 = \frac{\omega_2 \omega_3}{\omega_1}$$

(d) For this part, lift the restriction that the system is causal; it may or may not be. The expressions below are offered as candidate impulse responses for the system; the coefficients are all real and finite.

$$
\begin{aligned}
\times \quad h_I(t) &= \alpha_1\, e^{-\omega_1 t}\, u(t) + \alpha_2\, e^{-\omega_2 t}\, u(t) \\
\checkmark \quad h_{II}(t) &= \beta_1\, e^{-\omega_2 t}\, u(t) + \gamma_1\, e^{-\omega_3 t}\, u(t) \\
\checkmark \quad h_{III}(t) &= \beta_1\, e^{-\omega_2 t}\, u(-t) + \gamma_1\, e^{-\omega_3 t}\, u(t) \\
\times \quad h_{IV}(t) &= \beta_1\, e^{-\omega_2 t}\, u(t) + \gamma_1\, e^{-\omega_3 t}\, u(-t) \\
\checkmark \quad h_V(t) &= \beta_1\, e^{-\omega_2 t}\, u(-t) + \gamma_1\, e^{-\omega_3 t}\, u(-t)
\end{aligned}
$$

From the list above, choose every valid candidate impulse response for the system. Explain your reasoning succinctly, but clearly and convincingly.

The system has three possible regions of convergence

(•) $-\omega_2 < Re(s) \Rightarrow$ Each Pole contributes a causal term $\Rightarrow h_{II}(t)$ valid

(•) $-\omega_3 < Re(s) < -\omega_2 \Rightarrow$ The pole @ $-\omega_3$ contributes a causal term and the pole at $-\omega_2$ contributes an anti-causal term $\Rightarrow h_{III}(t)$ valid

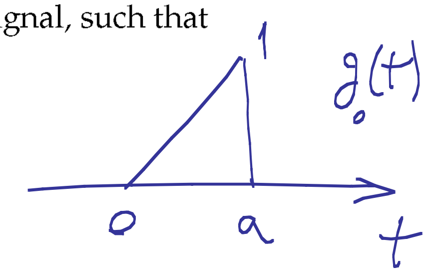
(•) $Re(s) < -\omega_3 \Rightarrow$ Each pole contributes an anti-causal term $\Rightarrow h_V(t)$ valid

(•) $h_I(t)$ is invalid because the zero at $-\omega_1$ cannot contribute to the impulse response.

(•) $h_{IV}(t)$ is invalid because its two terms have conflicting RoCs: $e^{-\omega_2 t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\omega_2}$ $-\omega_2 < Re(s)$, whereas $e^{-\omega_3 t}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s+\omega_3}$ $Re(s) < -\omega_3$

**MT3.3 (25 Points)** Let $g_0$ be a finite-duration continuous-time signal, such that

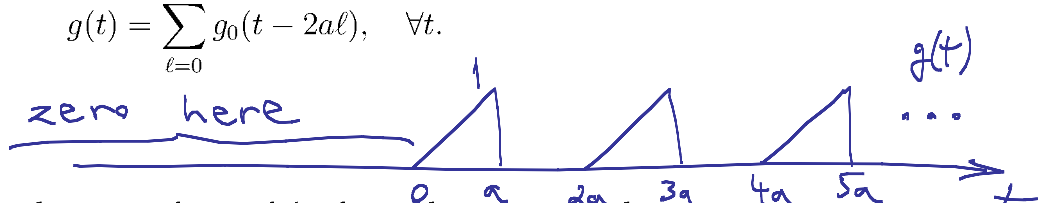
$$g_0(t) = \begin{cases} t/a & 0 < t < a \\ 0 & \text{elsewhere,} \end{cases}$$




where $a > 0$.

We construct a related signal $g$ as follows:

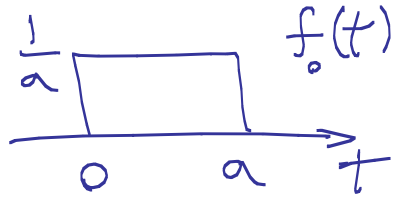
$$g(t) = \sum_{\ell=0}^{+\infty} g_0(t - 2a\ell), \quad \forall t.$$



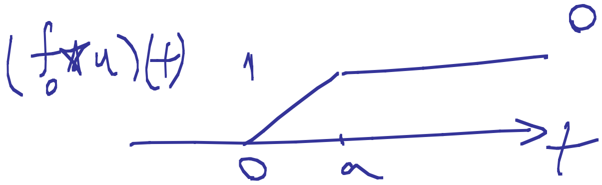

Determine $\widehat{G}_0(s)$, the Laplace transform of the finite-duration signal $g_0$.

Determine $\widehat{G}(s)$, the Laplace transform of $g$. Your answer should be in terms of $a$.

Let $f_0$ be




$\Longrightarrow (f_0 * u)(t) = \int_{-\infty}^{t} f_0(\tau)\, d\tau$ is




$\Longrightarrow g_0(t) = (f_0 * u)(t) - u(t-a) \Longrightarrow$

$\hat{G}_0(s) = \frac{\hat{F}_0(s)}{s} - \frac{e^{-as}}{s} = \frac{\hat{F}_0(s) - e^{-as}}{s}$. But $f_0(t) = \frac{1}{a}[u(t) - u(t-a)]$

$\Rightarrow \hat{F}_0(s) = \frac{1 - e^{-as}}{as} \Rightarrow \hat{G}_0(s) = \frac{\frac{1-e^{-as}}{as} - e^{-as}}{s} = \frac{1-(1+as)e^{-as}}{as^2}$ $\quad 0 < Re(s)$

$\hat{G}(s) = \sum_{\ell=0}^{\infty} \hat{G}_0(s) e^{-2as\ell} = \hat{G}_0(s) \sum_{\ell=0}^{\infty} (e^{-2as})^{\ell} = \frac{\hat{G}_0(s)}{1 - e^{-2as}} \Longrightarrow$

provided $\underbrace{|e^{-2as}| < 1}_{\text{i.e., } 0 < Re(s)}$

$$\hat{G}(s) = \frac{1 - (1+as)e^{-as}}{as^2(1 - e^{-2as})} \quad 0 < Re(s)$$

LAST Name Odic FIRST Name Pierre

Discussion Time Less discussion. More action.

| Problem Name | Points 10 | Your Score 10 |
|---|---|---|
| 1 | 40 | 40 |
| 2 | 40 | 40 |
| 3 | 25 | 25 |
| **Total** | **115** | 115 |