Exam 2

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<tr>
<th>Last name</th>
<th>First name</th>
<th>SID</th>
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</thead>
</table>

- You have 1 hour and 45 minutes to complete this exam.
- The exam is closed-book and closed-notes; calculators, computing and communication devices are not permitted.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- However, two handwritten and not photocopied double-sided sheet of notes is allowed.
- Additionally, you receive Tables 3.1, 3.2, 4.1, 4.2, 5.1, 5.2, 9.1, 9.2 from the class textbook.
- If we can’t read it, we can’t grade it.
- We can only give partial credit if you write out your derivations and reasoning in detail.
- You may use the back of the pages of the exam if you need more space.

*** Good Luck! ***

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points earned</th>
<th>out of</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td></td>
<td>29</td>
</tr>
<tr>
<td>Problem 2</td>
<td></td>
<td>28</td>
</tr>
<tr>
<td>Problem 3</td>
<td></td>
<td>27</td>
</tr>
<tr>
<td>Problem 4</td>
<td></td>
<td>33</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>117</td>
</tr>
</tbody>
</table>
Problem 1 (Short Questions.)  

(a) (4 Pts) For the system in Figure 1,

\[
H(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq \omega_0 \\
0, & \text{otherwise.}
\end{cases}
\]  

(1)

Sketch the frequency response \(G(j\omega)\) of the overall system between \(x(t)\) and \(y(t)\).

(b) (15 Pts) A causal LTI system is described by the following differential equation:

\[
\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + 2y(t) = \frac{d^2}{dt^2}x(t) - x(t)
\]  

(2)

Is this system stable? Does this system have a causal and stable inverse system?
Figure 2: Quadrature modulation.

Figure 3: “Improved” quadrature modulation.

(c) (10 Pts) As you have seen in the homework, “quadrature multiplexing” is the system shown in Figure 2, where

$$ H(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \leq \omega_M \\
0, & \text{otherwise}.
\end{cases} \quad \text{and} \quad G(j\omega) = \begin{cases} 
1, & \text{for } |\omega| \geq \omega_c \\
0, & \text{otherwise}.
\end{cases} \quad (3) $$

Both original signals are assumed to be bandlimited: $X(j\omega) = Y(j\omega) = 0$, for $|\omega| > \omega_M$; and the carrier frequency is $\omega_c > \omega_M$. The interesting feature is that the effective bandwidth of the signal $\tau(t)$ is only $2\omega_M$, the same as for a regular AM system with only the signal $x(t)$. Hence, $y(t)$ can ride along for free.

Now, your colleague remembers single-sideband AM and suggests to add the filters $G(j\omega)$ as shown in Figure 3. The effective bandwidth of the transmitted signal $\tilde{\tau}(t)$ is only $\omega_M$, half as much as in the original quadrature multiplexing system! Show that the “improved” system will not work. Hint: Find a pair of example spectra $X(j\omega)$ and $Y(j\omega)$ for which $R(j\omega)$ is not zero, but $\tilde{R}(j\omega) = 0$ for all $\omega$. Then, argue (in a few keywords) why this invalidates the “improved” quadrature modulation.
Problem 2 (Discrete-time processing of continuous-time signals.)

For the system in Figure 4,

$$H(j \omega) = \begin{cases} 1, & \text{for } |\omega| \leq \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad H_r(j \omega) = \begin{cases} T, & \text{for } |\omega| \leq \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

(a) (20 Pts) Give the formula for the overall system response $G(j \omega)$, relating $x(t)$ and $y(t)$. Also give a sketch of the magnitude $|G(j \omega)|$, paying particular attention to the labeling of the frequency axis. No derivation is necessary to get full credit.

(b) (8 Pts) For $x(t) = e^{j \pi t / (2T)}$, determine the corresponding output signal $y(t)$. Your answer should not contain an integral, but apart from that, there is no need to simplify it down.
Problem 3 (Sampling System Design.)

The signal \( x(t) \) has the Fourier transform shown in Figure 5.

![Figure 5:]

(a) (5 Pts) As a function of \( T \) (as in Figure 5), determine the smallest sampling frequency \( \omega_s = \frac{2\pi}{T_s} \) (where \( T_s \) is the sampling interval) for which perfect reconstruction can be guaranteed for the signal \( x(t) \). A graphical justification (sketch with labels on the frequency axis) is sufficient.

(b) (10 Pts) Consider the signal \( y(t) = h(t) \ast x(t) \), where \( h(t) \) is the impulse response of the filter \( H(j\omega) \) in Figure 5. Sketch the spectra of the two discrete-time signals

\[
\begin{align*}
    x[n] &= x(nT) \\
    y[n] &= y(nT),
\end{align*}
\]

where \( T \) is the same as in Figure 5. Which effect explains the difference between \( x[n] \) and \( y[n] \)?
(c) (12 Pts) The goal is now to implement a sampler with sampling interval $T_0 = T/2$, where $T$ is as in Figure 5. Unfortunately, such a fast sampler is not available in the current technology. Instead, you have access to the following devices:

- samplers with sampling interval $T$, where $T$ is the same as in Figure 5 (any number)
- anti-aliasing filters with the frequency response given in Figure 5 (any number)
- continuous-time signal adders/subtractors (any number)
- any discrete-time processing devices (ideal filters included).

Draw the block diagram of a system that takes as an input the signal $x(t)$ (with spectrum as shown in Figure 5) as outputs the discrete-time signal $x_0[n] = x(nT_0)$. Hint: To maximize your chance of partial credit, give spectral plots of intermediate signals in your system.
Problem 4 (PAM.)

Two pulses are suggested for a PAM system:

\[ q_1(t) = \begin{cases} 
1, & |t| \leq T/4 \\
0, & \text{otherwise} 
\end{cases} \quad \text{and} \quad q_2(t) = \begin{cases} 
-1, & -T/4 \leq t < 0 \\
1, & 0 \leq t \leq T/4 \\
0, & \text{otherwise} 
\end{cases} \]

The PAM signal is then

\[ x_m(t) = \sum_{n=-\infty}^{\infty} s[n]q_m(t - nT), \quad \text{for} \ m = 1, 2. \quad (7) \]

Throughout this problem, we assume that the data signal is merely \( s[n] = 1 \), for all \( n \).

(a) (6 Pts) Find the powers \( P_1 \) and \( P_2 \) of the two PAM signals \( x_1(t) \) and \( x_2(t) \).

(b) (10 Pts) Give the formula for the Fourier series coefficients of the signal \( x_1(t) \), and explicitly evaluate the coefficients \( a_0, a_1 \) and \( a_{-1} \). Then, do the same for the signal \( x_2(t) \).
(c) (7 Pts) To actually transmit our PAM signal, we first low-pass filter it:

\[ \tilde{x}_m(t) = h(t) \ast x_m(t), \text{ where } H(j\omega) = \begin{cases} 1, & \text{for } |\omega| \leq \frac{10\pi}{T} \\ 0, & \text{otherwise.} \end{cases} \]  

Then, we transmit the signals \( y_1(t) = \tilde{x}_1(t) \cos(\frac{40\pi}{T}t) \) and \( y_2(t) = \tilde{x}_2(t) \cos(\frac{40\pi}{T}t) \). Sketch the Fourier transforms of these two signals in the plots provided below, carefully labeling the frequency axis. In the magnitude plots (i.e., \(|Y_1(j\omega)|\) and \(|Y_2(j\omega)|\), respectively), the amplitudes need not be exact. **Remark:** The current labels on the frequency axis in the plots are for your convenience only. If you prefer, you can cross them out and start from scratch.

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Figure 6:
(d) (10 Pts) The communication channel’s effect on the signal can be described by the following band-pass filter:

\[
H_{\text{channel}}(j\omega) = \begin{cases} 
\sin^2(\omega T/4 - \pi) & \text{for } 36\pi/T < |\omega| < 38\pi/T \\
1 & \text{for } 38\pi/T \leq |\omega| \leq 42\pi/T \\
\sin^2(\omega T/4 - \pi) & \text{for } 42\pi/T < |\omega| < 44\pi/T \\
0, & \text{otherwise.}
\end{cases}
\]  

(9)

The channel output signal is then \( z_1(t) = y_1(t) \ast h_{\text{channel}}(t) \) and \( z_2(t) = y_2(t) \ast h_{\text{channel}}(t) \), respectively. Assuming that \( s[n] = 1 \), for all \( n \), find the power of \( z_1(t) \) and \( z_2(t) \). These are the received powers. Which pulse is more efficient for transmission across this channel?