EE120, Fall 2005, Midterm 1, Gastpar

Problem 1 (Properties of Systems.) 26 Points

(a) (4 Pts) True or false: A BIBO stable, causal, linear system must be time-invariant. Give a proof if you believe it’s true. Give a counterexample if you believe it’s false.

(b) (6 Pts) True or false: If the (discrete-time) input to a linear, time-invariant system is periodic with period N, then its output must also be periodic with period N. Give a proof if you believe it’s true. Give a counterexample if you believe it’s false.

(c) (16 Pts) A system with input $x(t)$ and output $y(t)$ can be described as follows:

$$y(t) = \left(\int_{-\infty}^{0} e^{\tau_1} x(t-\tau_1) d\tau_1 \right)^{\left(\int_{-\infty}^{\infty} e^{-|\tau_2 - t|} x(\tau_2) d\tau_2 \right)} \quad (4)$$

Is this a BIBO stable system? Carefully develop your argument.

Problem 2 (Convolution.) 20 Points

Let

$$x(t) = \begin{cases} -t, & -1 \le t < 0, \\ 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

and

$$y(t) = \begin{cases} 1, & |t| \le 1/2 \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Sketch $x(t) * y(t)$, carefully labeling both axes. Note: The correct plot with the correct labeling on both axes will get full credit; there is no requirement to write down formulas. We can compute the convolution of $x(t)$ and $y(t)$ directly, or use linearity to compute two simpler convolutions, $x(t) * y(t) = (x_1(t) + x_2(t)) * y(t) = x_1(t) * y(t) + x_2(t) * y(t)$.

Problem 3 (Short-time Fourier Transform.) 22 Points

To compute a Fourier transform requires us to know the entire signal $x(t)$, for $-\text{infinity} < t < \text{infinity}$. In practice, when doing spectral analysis, we cannot usually wait that long. One way out of this dilemma is to perform what people often call the short-time Fourier transform: Instead of looking at the actual Fourier transform of $x(t)$, we consider

$$x_w(t) = x(t)w(t), \tag{11}$$

where $w(t)$ is sometimes called a "window". The short-time Fourier transform of $x(t)$ with respect to the window $w(t)$ is then simply $X_w(jw)$, the Fourier transform of $x_w(t)$.

(a) (16 Pts) An obvious first choice of a window is simply to cut the signal $x(t)$ at some point,
i.e.,

$$w(t) = \begin{cases} 1, & |t| \leq T \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

For the special case $x(t) = \cos(w_0 t)$ find the formula for the short-term Fourier transform $X_w(jw)$. Give a sketch of $X_w(jw)$. Use $T = 5\pi/w_0$ for the sketch.

(b) (6 Pts) Explain what happens as T and w_0 are varied. You may use sketches to do this.

Hint: Consider the "extreme" cases, T becomes very large or very small.

Problem 4 (System.) 29 Points

Consider the causal, linear, time-invariant discrete-time system that obeys the following difference equation:

$$y[n] - ay[n-2] = x[n] - bx[n-1] \tag{13}$$

(a) (6 Pts) Give a system diagram, using only multipliers, adders, and delay elements.

(b) (7 Pts) Find the frequency response of this system.

(c) (8 Pts) Find the impulse response of this system.

(d) (8 Pts) For what values of a and b is the system stable?

Problem 5 *(Non-linear System.)* 20 Points

A simple non-linear system is characterized by

$$y(t) = \left(\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \right)^2, \quad \text{where} \quad h(t) = 2\mathrm{sinc}(2t) \tag{18}$$

We would like to approximate this system by a linear and time-invariant system. The following two systems have been suggested. The first suggested LTI system has impulse response

$$h_1(t) \quad = \quad h(t). \tag{19}$$

and we denote its output by $\tilde{y}_1(t)$. The second suggested LTI system has frequency response

$$H_2(j\omega) \quad = \quad \begin{cases} 2 - |\omega|/(2\pi), & |\omega| \leq 2\pi \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

and we denote its output by $\tilde{y}_2(t)$.

(a) (15 Pts) For the special input signal $x(t) = 3\mathrm{sinc}(3t)$, find the errors

$$E_m \quad = \quad \int_{-\infty}^{\infty} (y(t) - \tilde{y}_m(t))^2 dt, \text{ for } m = 1, 2. \tag{21}$$

Which system incurs a smaller error, $h_1(t)$ or $h_2(t)$?

(b) (5 Pts) Repeat Part (a) for the input signal $x(t) = \cos(\frac{2}{3}\pi t)$, this time finding the power of the error signal $y(t) - \tilde{y}_m(t)$, for $m = 1, 2$. In this case, which system incurs a smaller error, $h_1(t)$ or $h_2(t)$?