EECS 120 **Fall 2004**

Midterm 2 Solutions

- The exam is for one hour and 50 minutes.

- The maximum score is 100 points. The maximum score for each part of each problem is indicated.

- The exam is closed-book and closed-notes. Calculators, computing, and communication devices are NOT permitted.

- Four double-sided sheets of notes are allowed. These should be legible to normal eyesight, i.e. the lettering should not be excessively small.

- No form of collaboration between students is allowed.

1. *(8 points)* State whether the following are true or false. In each case, give a brief explanation. A correct answer with a correct explanation gets 4 points. A correct answer without a correct explanation gets 1 points. A wrong answer gets 0 points.

   (a) If two signals $x(t)$ and $y(t)$ have the same *unilateral* Laplace transform, then the signals are identical.

   Solution : **False**. The unilateral Laplace transform of a signal $x(t)$ is defined as

   $$\mathcal{X}(s) = \int_{0-}^{\infty} x(t)e^{-st}dt \ .$$

   Thus, for example, any signal $x(t)$ with $x(t) = 0$ for all $t \geq 0$ will have $\mathcal{X}(s) = 0$.

   (b) Let the signal $x_1(t)$ have Laplace transform $X_1(s)$ with region of convergence R_1, and let the signal $x_2(t)$ have Laplace transform $X_2(s)$ with region of convergence R_2. Assume that $R_1 \cap R_2$ is not the empty set. Then the signal $y(t) = x_1(t) * x_2(t)$, which is the convolution of $x_1(t)$ with $x_2(t)$, has Laplace transform $Y(s) = X_1(s)X_2(s)$ with region of convergence equal to $R_1 \cap R_2$.

   Solution : **False**. In general all we can say is that the region of convergence of the Laplace transform of $y(t)$ contains $R_1 \cap R_2$, but it can be larger. For an example, consider $X_1(s) = \frac{1}{s+2}$ with ROC $Re(s) > -2$ and $X_2(s) = \frac{s+2}{(s+1)^2}$ with ROC $Re(s) < -1$. Then $y(t) = x_1(t) * x_2(t)$ has $Y(s) = \frac{1}{(s+1)^2}$ with ROC $Re(s) < -1$.

2. *(6 + 2 + 6 + 2 + 4 + 5 points)* Consider the impulse train

   $$p(t) = \sum_{n=-\infty}^{\infty} [\delta(t - 3n\Delta) + \delta(t - 3n\Delta - \Delta) - \delta(t - 3n\Delta - 2\Delta)] \ .$$

   Note carefully that every third impulse has sign opposite to that of the preceding two impulses.

Let $x(t)$ be a signal. We assume that $x(t)$ is band limited to $(-\omega_M, \omega_M)$, i.e.

$$X(j\omega) = 0 \ , \quad \text{if } |\ \omega\ | \geq \omega_M \ ,$$

where $X(j\omega)$ denotes the Fourier transform of $x(t)$.

Let

$$x_p(t) = x(t)p(t) \ .$$

Let $H(j\omega)$ be given by

$$H(j\omega) = \begin{cases} A & \text{if } -\omega_f < \omega < \omega_f \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus the filter with frequency response $H(j\omega)$ is a low pass filter which passes only frequencies in the range $-\omega_f < \omega < \omega_f$, with amplification A in the pass band.

Let $y(t)$ denote the output of this low pass filter when the input is $x_p(t)$.

(a) Let $P(j\omega)$ denote the Fourier transform of $p(t)$. Determine $P(j\omega)$.

(b) Choose some $X(j\omega)$ that is non-zero over $(-\omega_M, \omega_M)$ and has linear phase over $(-\omega_M, \omega_M)$, with the phase being non-zero except at $\omega = 0$. Sketch the magnitude and phase plots for the $X(j\omega)$ that you picked. Make sure to label all the important frequency, magnitude, and phase coordinates.

(c) Let $X_p(j\omega)$ denote the Fourier transform of $x_p(t)$. Sketch the magnitude and phase plots for $X_p(j\omega)$ corresponding to the $X(j\omega)$ that you picked, assuming the appropriate no-aliasing condition. Make sure to label all the important frequency, magnitude, and phase coordinates.

(d) What is the appropriate no-aliasing condition in the preceding part of the problem ?

(e) Find the conditions on ω_M, Δ, ω_f, and A which ensure that $y(t) = x(t)$ for all $x(t)$ that are band limited to $(-\omega_M, \omega_M)$.

(f) Is there a way to recover $x(t)$ from $x_p(t)$ even when the no-aliasing condition of part (d) does not hold ? Explain your answer.

Solution :

(a) Let $\omega_s = \frac{2\pi}{3\Delta}$. We have

$$\begin{aligned} P(j\omega) &= \frac{2\pi}{3\Delta} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) + \frac{2\pi}{3\Delta} e^{-j\omega\Delta} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) - \frac{2\pi}{3\Delta} e^{-2j\omega\Delta} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\ &= \frac{2\pi}{3\Delta} \sum_{k=-\infty}^{\infty} \left[1 + e^{-j\omega\Delta} - e^{-2j\omega\Delta}\right] \delta(\omega - k\omega_s) \\ &= \frac{2\pi}{3\Delta} \sum_{k=-\infty}^{\infty} \left[1 + e^{-jk\omega_s\Delta} - e^{-2jk\omega_s\Delta}\right] \delta(\omega - k\omega_s) \end{aligned}$$

$$= \frac{2\pi}{3\Delta} \sum_{k=-\infty}^{\infty} \left[1 + e^{-jk\frac{2\pi}{3}} - e^{-jk\frac{4\pi}{3}} \right] \delta(\omega - k\omega_s)$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_s)$$

where

$$a_k = \begin{cases} \frac{2\pi}{3\Delta} & \text{if } k \text{ is an integer multiple of 3} \\ \frac{2\pi}{3\Delta}(1+j) & \text{if } k-1 \text{ is an integer multiple of 3} \\ \frac{2\pi}{3\Delta}(1-j) & \text{if } k-2 \text{ is an integer multiple of 3} \end{cases}$$

(b) Choose some $X(j\omega)$ that is non-zero over $(-\omega_M, \omega_M)$, zero elsewhere, and has linear phase over $(-\omega_M, \omega_M)$, with the phase non-zero except at $\omega = 0$.

(c)

$$X_p(j\omega) = \frac{1}{2\pi} [X(j\omega) * P(j\omega)]$$
$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k X(j(\omega - k\omega_s))$$

A sketch of this is non-zero over regions of width $2\omega_M$ centered at the frequencies $k\omega_s$, $k \in \mathbf{Z}$, with the regions non-overlapping iff $2\omega_M \leq \omega_s$. Note that the magnitude is $\frac{1}{3\Delta}$ times the magnitude of $X(j\omega)$ and the phase is identical to that of $X(j\omega)$ in the regions centered at $k\omega_s$ for k a multiple of 3. The magnitude is $\sqrt{2}\frac{1}{3\Delta}$ times the magnitude of $X(j\omega)$ and the phase is $\frac{\pi}{4}$ plus the phase of $X(j\omega)$ in the regions centered at $k\omega_s$ for $k-1$ a multiple of 3. The magnitude is $\sqrt{2}\frac{1}{3\Delta}$ times the magnitude of $X(j\omega)$ and the phase is $-\frac{\pi}{4}$ plus the phase of $X(j\omega)$ in the regions centered at $k\omega_s$ for $k-2$ a multiple of 3.

(d) The no-aliasing condition is $2\omega_M \leq \omega_s$, i.e.

$$\Delta \leq \frac{\pi}{3\omega_M} .$$

(e) We need the no-aliasing condition to make sure that there is no loss of the spectral shape of $X(j\omega)$ when we go to $X_p(j\omega)$. We need $\omega_M \leq \omega_f \leq \omega_s - \omega_M$ to ensure that the low pass filter $H(j\omega)$ will only pick up the portion of the spectrum of $X_p(j\omega)$ that lies in the interval of width $2\omega_M$ centered at $\omega = 0$. We need $A = \frac{2\pi}{a_0} = 3\Delta$ to ensure that the output of the filter is $x(t)$ rather than just a scaled version of $x(t)$.

(f) Some thought reveals that it is only necessary to have

$$\Delta \leq \frac{\pi}{\omega_M}$$

in order to be able to recover $x(t)$ from $x_p(t)$. This is because it is possible in principle to recover the samples $x(n\Delta)$, $n \in \mathbf{Z}$, from $x_p(t)$, so $x_p(t)$ contains all the information we would have got from periodic sampling of $x(t)$ with period Δ. In fact it contains precisely this information. The Nyquist sampling criterion tells us that as long as $2\omega_M \leq \frac{2\pi}{\Delta}$, we can recover the signal $x(t)$ from these samples, hence from $x_p(t)$.

3. *(12 points)*

Let $x(t)$ be a low pass signal that is bandlimited to $(-\omega_M, \omega_M)$, i.e. if $X(j\omega)$ denotes the Fourier transform of $x(t)$ then we have

$$X(j\omega) = 0 \text{ if } |\omega| \geq \omega_M \ .$$

Let $y(t) = A_c x(t)\cos(\omega_c t)$ denote the DSB-SC signal resulting from amplitude modulation of $x(t)$ onto the carrier $A_c \cos(\omega_c t)$. Assume that $\omega_c >> \omega_M$.

Let $p(t)$ denote the periodic signal with period $T_c = \frac{2\pi}{\omega_c}$ given by

$$p(t) = \begin{cases} 1 & \text{if } |t - nT_c| \leq \frac{T_c}{4} \text{ for some } n \in \mathbf{Z} \\ 0 & \text{otherwise} \end{cases}$$

Let $z(t) = y(t)p(t)$. The signal $z(t)$ is passed through an ideal low pass filter $H(j\omega)$ with cutoff frequency ω_M, i.e.

$$H(j\omega) = \begin{cases} 1 & \text{if } |\omega| \leq \omega_M \\ 0 & \text{otherwise} \end{cases} .$$

Let the output of this filter be denoted $w(t)$.

Determine $w(t)$.

Solution :

$p(t)$ is periodic with period $T_c = \frac{2\pi}{\omega_c}$. We may therefore write a Fourier series expansion for it :

$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_c t} ,$$

where

$$\begin{aligned} a_0 &= \frac{1}{2} \text{ and} \\ a_k &= \frac{1}{2}\text{sinc}(\frac{k}{2}) \text{ if } k \neq 0 \ . \end{aligned}$$

We now write

$$z(t) = y(t)p(t) = A_c x(t)\cos(\omega_c t)\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_c t}\right] .$$

We write

$$\cos(\omega_c t) = \frac{1}{2}\left[e^{j\omega_c t} + e^{-j\omega_c t}\right] .$$

Since $z(t)$ is to be sent through the ideal low pass filter $H(j\omega)$, and since we assumed that $\omega_M << \omega_c$, to produce $w(t)$, we see that

$$w(t) = A_c x(t)\left[\frac{1}{2}a_1 + \frac{1}{2}a_{-1}\right] .$$

Finally, observing that $a_1 = a_{-1} = \frac{1}{\pi}$, we have

$$w(t) = \frac{A_c}{\pi} x(t) .$$

4. *(2 + 6 + 6 points)*

Let $x[n]$ be a discrete time signal whose discrete time Fourier transform (DTFT) $X(e^{j\omega})$ is bandlimited to $(-\frac{\pi}{10}, \frac{\pi}{10})$, i.e.

$$X(e^{j\omega}) = 0 \text{ if } |\, \omega \,| \leq \pi \text{ and } |\, \omega \,| \geq \frac{\pi}{10} \ .$$

Recall that

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

is a periodic function of ω with period 2π, which is why the extra condition that $|\, \omega \,| \leq \pi$ is needed in the notion of "bandlimited".

Let $z[n]$ be created from $x[n]$ by dropping all terms for n an integer multiple of 4, i.e.

$$z[n] = \begin{cases} 0 & \text{if } n \text{ is a multiple of } 4 \\ x[n] & \text{otherwise} \ . \end{cases}$$

You can think of $z[n]$ as $x[n] - y[n]$, where $y[n]$ is the discrete time sampling of $x[n]$ with period 4, if you like.

Let $w[n]$ denote the up-sampled version of $z[n]$ with up-sampling factor 3, i.e. to get $w[n]$ we first create

$$v[n] = \begin{cases} z[\frac{n}{3}] & \text{if } n \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases} ,$$

and then pass $v[n]$ through an ideal discrete time low pass filter that is bandlimited to $(-\frac{\pi}{3}, \frac{\pi}{3})$.

For the following, pick some $X(e^{j\omega})$ that is periodic with period 2π, non-zero over $(-\frac{\pi}{10}, \frac{\pi}{10})$, and has linear phase over $(-\frac{\pi}{10}, \frac{\pi}{10})$ which is non-zero except at $\omega = 0$.

(a) Sketch the magnitude and phase plots for the $X(e^{j\omega})$ you picked. Make sure to label all the important frequency, magnitude, and phase coordinates.

(b) Let $Z(e^{j\omega})$ denote the DTFT of $z[n]$. Sketch the magnitude and phase plots of $Z(e^{j\omega})$ corresponding to the $X(e^{j\omega})$ that you picked. Make sure to label all the important frequency, magnitude, and phase coordinates.

(c) Let $W(e^{j\omega})$ denote the DTFT of $w[n]$. Sketch the magnitude and phase plots of $W(e^{j\omega})$ corresponding to the $X(e^{j\omega})$ that you picked. Make sure to label all the important frequency, magnitude, and phase coordinates.

Solution :

(a) Pick some $X(e^{j\omega})$ that is periodic with period 2π, non-zero over $(-\frac{\pi}{10}, \frac{\pi}{10})$, and has linear phase over $(-\frac{\pi}{10}, \frac{\pi}{10})$ which is non-zero except at $\omega = 0$.

(b) Let

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k] \ .$$

This has DTFT :

$$P(e^{j\omega}) = \frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)\ ,$$

where $\omega_s = \frac{2\pi}{4} = \frac{\pi}{2}$.
Let the periodic sampling of $x[n]$ with period 4 be denoted by $y[n]$. Then

$$y[n] = x[n]p[n]\ ,$$

so $y[n]$ has DTFT $Y(e^{j\omega})$ given by :

$$Y(e^{j\omega}) = \frac{1}{2\pi}X(e^{j\omega}) *_{2\pi} P(e^{j\omega})\ ,$$

where $*_{2\pi}$ denotes circular convolution. It follows that

$$Y(e^{j\omega}) = \frac{1}{4}\sum_{k=0}^{3}X(e^{j(\omega-k\frac{\pi}{2})})\ .$$

Since

$$Z(e^{j\omega}) = X(e^{j\omega}) - Y(e^{j\omega})\ ,$$

a sketch of $Z(e^{j\omega})$ is periodic with period 2π, and is non-zero in intervals of length $\frac{2\pi}{10}$ centered at the points $k\frac{\pi}{2}$, $k \in \mathbf{Z}$. The magnitude in the intervals centered around $k2\pi$, $k \in \mathbf{Z}$, is $\frac{3}{4}$ the magnitude of $X(e^{j\omega})$, and the phase is the same as that of $X(e^{j\omega})$. The magnitude in the intervals centered around $k\frac{\pi}{2}$, $k \in \mathbf{Z}$, k not a multiple of 4, is $\frac{1}{4}$ the magnitude of $X(e^{j\omega})$, and the phase is π plus that of $X(e^{j\omega})$.

(c) Let $V(e^{j\omega})$ denote the DTFT of $v[n]$. Then

$$V(e^{j\omega}) = Z(e^{j3\omega})\ .$$

$V(e^{j\omega})$ is nonzero in intervals of length $\frac{2\pi}{30}$ centered at $k\frac{\pi}{6}$, $k \in \mathbf{Z}$. Its magnitude and phase plots look like a compressed version (by a factor of 3 on the frequency axis) of the corresponding plots of $Z(e^{j\omega})$. Finally, the plots for $W(e^{j\omega})$ are got from those of $V(e^{j\omega})$ be erasing everything that does not lie in the intervals of length $\frac{2\pi}{3}$ centered at the frequencies $k2\pi$, $k \in \mathbf{Z}$.

5. *(10 points)*

Let $x[n]$ be a discrete time signal. Assume that its discrete time Fourier transform (DTFT) $X(e^{j\omega})$, which is a periodic function with period 2π, is band limited to $(-\omega_M, \omega_M)$, i.e. assume that

$$X(e^{j\omega}) = 0\ , \quad \text{if } \omega_M <\mid \omega \mid\leq \pi\ .$$

The extra condition that $\mid \omega \mid\leq \pi$ is needed in the notion of "bandlimited" because $X(e^{j\omega})$, is periodic with period 2π. We assume, of course, that $\omega_M < \pi$.

Assume that $x[n] \neq 0$ for all n.

Let $y[n]$ be a discrete time signal with DTFT $Y(e^{j\omega})$. Assume that this is also bandlimited to $(-\omega_M, \omega_M)$.

Let $a > 1$ be a real number such that $a\omega_M < \pi$, and let

$$z[n] = \cos(a\ \omega_M n) + y[n]\ .$$

Let $v[n] = x[n]z[n]$. The signal $v[n]$ is periodically sampled with period N. Here $N \geq 1$ is some integer.

For what values of a and N is it possible to recover *both* $x[n]$ and $y[n]$ from the samples $v[nN], n \in \mathbf{Z}$?

Solution :

Let $Z(e^{j\omega})$ denote the DTFT of $z[n]$. Then $Z(e^{j\omega})$ is periodic with period 2π, and

$$Z(e^{j\omega}) = \sum_{l=-\infty}^{\infty} [\pi\delta(\omega - a\omega_M - 2\pi l) + \pi\delta(\omega + a\omega_M - 2\pi l)] + Y(e^{j\omega})\ ,$$

where $Y(e^{j\omega})$ denotes the DTFT of $y[n]$. Now, $V(e^{j\omega})$, the DTFT of $v[n]$, is the circular convolution of $X(e^{j\omega})$, the DTFT of $x[n]$, with $Z(e^{j\omega})$.

We see that if we want to have no overlap between the terms coming from the circular convolution of $X(e^{j\omega})$ and $Y(e^{j\omega})$ and the terms coming from the circular convolution of $X(e^{j\omega})$ with $\sum_{l=-\infty}^{\infty} [\pi\delta(\omega - a\omega_M - 2\pi l) + \pi\delta(\omega + a\omega_M - 2\pi l)]$, we must have

$$a\omega_M - \omega_M \geq 2\omega_M\ , \quad \text{i. e.} a \geq 3\ .$$

If we want to have no aliasing between the terms coming from the circular convolution of $X(e^{j\omega})$ with $\sum_{l=-\infty}^{\infty} [\pi\delta(\omega - a\omega_M - 2\pi l) + \pi\delta(\omega + a\omega_M - 2\pi l)]$, we must have

$$a\omega_M + \omega_M \leq \pi\ .$$

Further, in order to be able to recover $V(e^{j\omega})$ from the samples $v[nN], n \in \mathbf{Z}$ we must have

$$a\omega_M + \omega_M \leq \frac{\pi}{N}\ .$$

Thus, the conditions

$$a \geq 3 \text{ and } N(a+1) \leq \frac{\pi}{\omega_M}$$

would allow us to recover both $x[n]$ and $y[n]$ from the samples $v[nN], n \in \mathbf{Z}$, because we could first recover $V(e^{j\omega})$ and from that recover the circular convolution of $X(e^{j\omega})$ with $\sum_{l=-\infty}^{\infty} [\pi\delta(\omega - a\omega_M - 2\pi l) + \pi\delta(\omega + a\omega_M - 2\pi l)]$, which tells us what $X(e^{j\omega})$ is, and hence what the sequence $x[n]$ is. Also, from $V(e^{j\omega})$ we can recover the circular convolution of $X(e^{j\omega})$ and $Y(e^{j\omega})$, which tells us what the sequence $x[n]y[n]$ is, and since we now know what the sequence $x[n]$ is, and because $x[n] \neq 0$ for all n, we can find out what the sequence $y[n]$ is.

It is possible to be more clever than this. It is not necessary to have no overlap between the terms coming from the circular convolution of $X(e^{j\omega})$ and $Y(e^{j\omega})$ and the terms coming from the circular convolution of $X(e^{j\omega})$ with $\sum_{l=-\infty}^{\infty} [\pi\delta(\omega - a\omega_M - 2\pi l) + \pi\delta(\omega + a\omega_M - 2\pi l)]$. Indeed, the conditions

$$a \geq 2 \text{ and } N(a+1) \leq \frac{\pi}{\omega_M}$$

are also okay, because then we could first recover $V(e^{j\omega})$ as before, then recover $X(e^{j\omega})$ from the *upper sidebands* of the terms coming from the circular convolution of $X(e^{j\omega})$ with $\sum_{l=-\infty}^{\infty} [\pi\delta(\omega - a\omega_M - 2\pi l) + \pi\delta(\omega + a\omega_M - 2\pi l)]$, which tells us what the sequence $x[n]$ is, use this knowledge to compensate for the overlap with the terms coming from the circular convolution of $X(e^{j\omega})$ and $Y(e^{j\omega})$, which then tells us what the sequence $x[n]y[n]$ is, and finally tells us what the sequence $y[n]$ is, as before.

6. *(2 + 2 + 9 points)*

A causal LTI system has system function

$$H(s) = \frac{s-4}{s^2+5s+4} \ .$$

(a) Determine the region of convergence of the system function.

(b) Is the system stable ? Why ?

(c) Determine the output of the system when the input is

$$x(t) = e^{-2|t|} \ .$$

Solution :

(a) The poles of the system function are at -4 and -1, and it has a finite zero at 4. Since the system is causal, the ROC of the system function is $Re(s) > -1$.

(b) Since the poles of the system function are strictly in the left half plane, and because we are given that the system is causal, it is stable.

(c) The input can be written as

$$x(t) = e^{-2t}u(t) + e^{2t}u(-t) \ .$$

The Laplace transform of the input is

$$\begin{aligned} X(s) &= \frac{1}{s+2} - \frac{1}{s-2} \\ &= \frac{-4}{s^2-4} \ , \quad -2 < Re(s) < 2 \ . \end{aligned}$$

Let $y(t)$ denote the output of the system when the input is $x(t)$. The Laplace transform of the output, $Y(s)$ is then $\frac{-4(s-4)}{(s^2-4)(s^2+5s+4)}$ with ROC containing $-1 < Re(s) < 2$. Since the poles of this rational function are at -4, -2, -1, and 2, the ROC must be $-1 < Re(s) < 2$.

We use the technique of partial fraction expansion to write

$$Y(s) = \frac{a}{s+4} + \frac{b}{s+2} + \frac{c}{s+1} + \frac{d}{s-2} \ .$$

The coefficients can be found as

$$\begin{aligned} a &= (s+4)Y(s)\mid_{s=-4} = \frac{-4(-4-4)}{(16-4)(-4+1)} = -\frac{8}{9} \\ b &= (s+2)Y(s)\mid_{s=-2} = \frac{-4(-2-4)}{(-2-2)(4-10+4)} = 3 \\ c &= (s+1)Y(s)\mid_{s=-1} = \frac{-4(-1-4)}{(1-4)(-1+4)} = -\frac{20}{9} \\ c &= (s-2)Y(s)\mid_{s=2} = \frac{-4(2-4)}{(2+2)(4+10+4)} = \frac{1}{9} . \end{aligned}$$

From this, based on the ROC, we conclude that

$$y(t) = -\frac{8}{9}e^{-4t}u(t) + 3e^{-2t}u(t) - \frac{20}{9}e^{-t}u(t) - \frac{1}{9}e^{2t}u(-t) .$$

7. *(18 points)*

Each of the plots A, B, C, and D on the next two pages is a pair, giving $\mid H(j\omega) \mid$ as a function of $\omega > 0$ in radians on a log-log plot and $\angle H(j\omega)$ in degrees as a function of $\omega > 0$ in radians on a linear-log plot. These plots correspond, in some unknown order, to the frequency responses of the causal stable LTI systems with the following system functions (the finite poles and zeros of these system functions are also explicitly given for convenience).

Number	System function	poles	zeros
1.	$\frac{1}{s^2+2s+5}$	$-1+j2,\ -1-j2$	none
2.	$\frac{s^2+4s+8}{s^2+2s+5}$	$-1+j2,\ -1-j2$	$-2+j2,\ -2-j2$
3.	$\frac{s^2-8s+20}{s^2+2s+5}$	$-1+j2,\ -1-j2$	$4+j2,\ 4-j2$
4.	$\frac{s^2+8s+20}{s^2+2s+5}$	$-1+j2,\ -1-j2$	$-4+j2,\ -4-j2$

Match up the system functions with the corresponding plots. Every correct answer gets 3 points, and every wrong answer gets -1 points. Also, explain your reasoning for why you matched the system functions to the plots the way you did. 6 points are allocated to evaluate your reasoning.

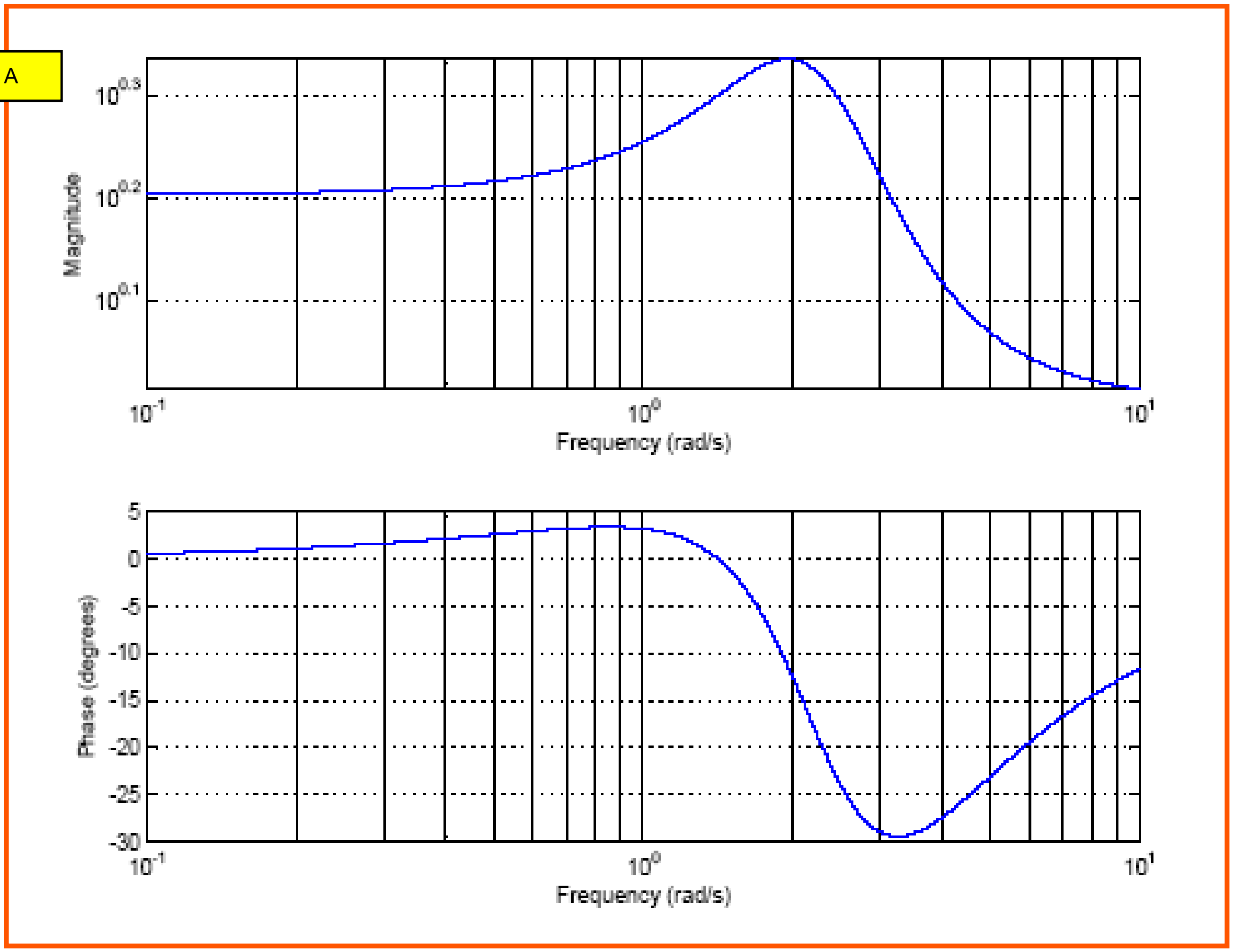
Note : Even though negative points will be given for wrong answers, the lowest score possible on this problem as a whole will be zero points.

Solution :

$$
\begin{aligned}
1. &\quad \Leftrightarrow \quad \text{Plot D} \\
2. &\quad \Leftrightarrow \quad \text{Plot A} \\
3. &\quad \Leftrightarrow \quad \text{Plot C} \\
4. &\quad \Leftrightarrow \quad \text{Plot B}
\end{aligned}
$$

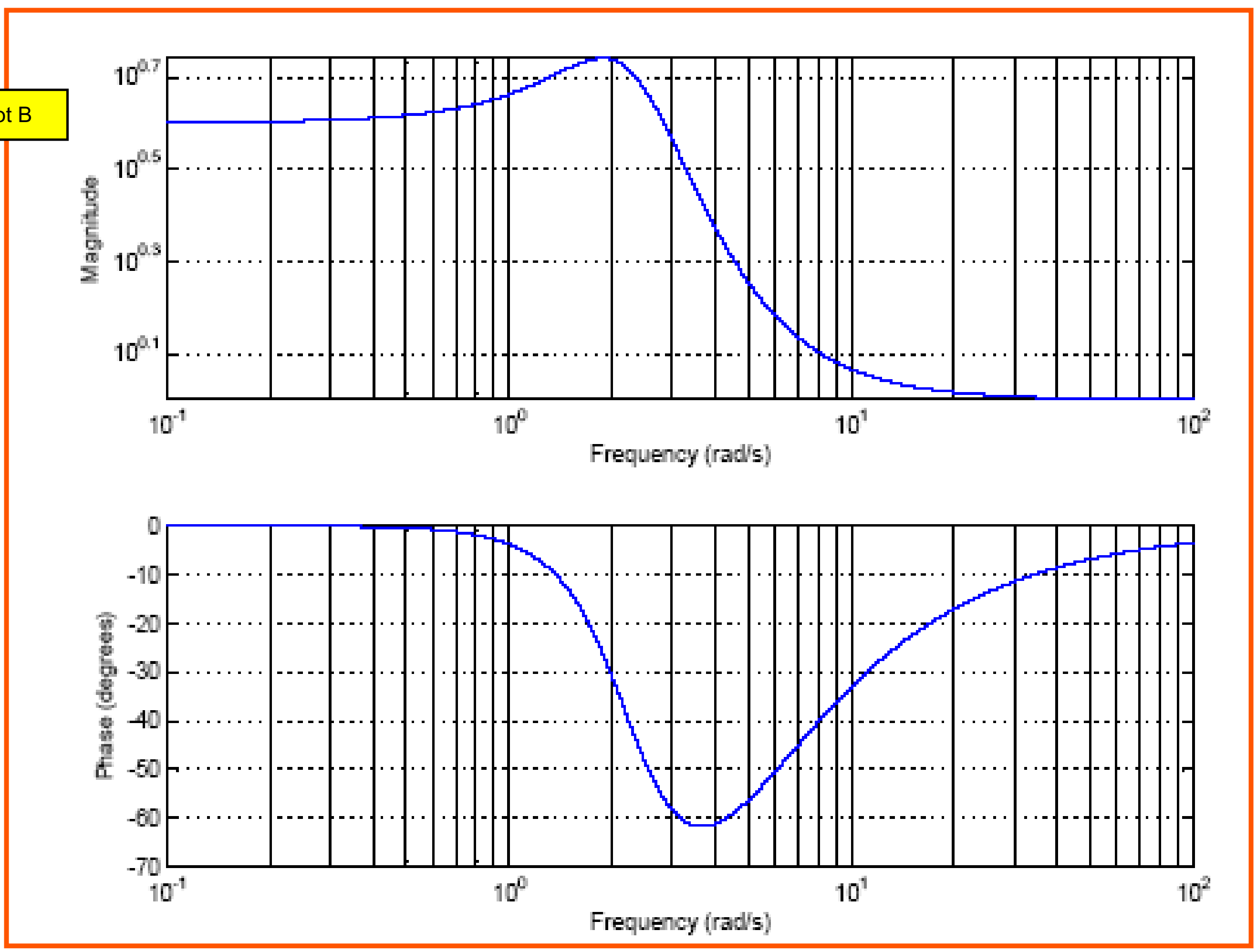
The only plot which has phase going to -180^o as $\omega \to \infty$ is plot D, so this must be the one corresponding to the system function 1 (notice that each zero contributes an angle of $+90^o$ and each pole contributes an angle of -90^o as $\omega \to \infty$). Of the remaining three plots, the one with the least magnitude at $\omega = 0$ is plot A, so this must be the one corresponding to system function 2 (notice that of these three system functions the zeros are closest to the imaginary axis in system function 2). The remaining two plots have identical magnitude functions. However, for positive frequencies, the absolute value of the phase of the system function 4 can never exceed -180^o, as it does in plot C (because each zero contributes less phase than the corresponding pole removes and each pole removes at most a phase of 90^o) so system function 4 must correspond to plot B, which leaves system function 3 corresponding to plot C.

Plot A



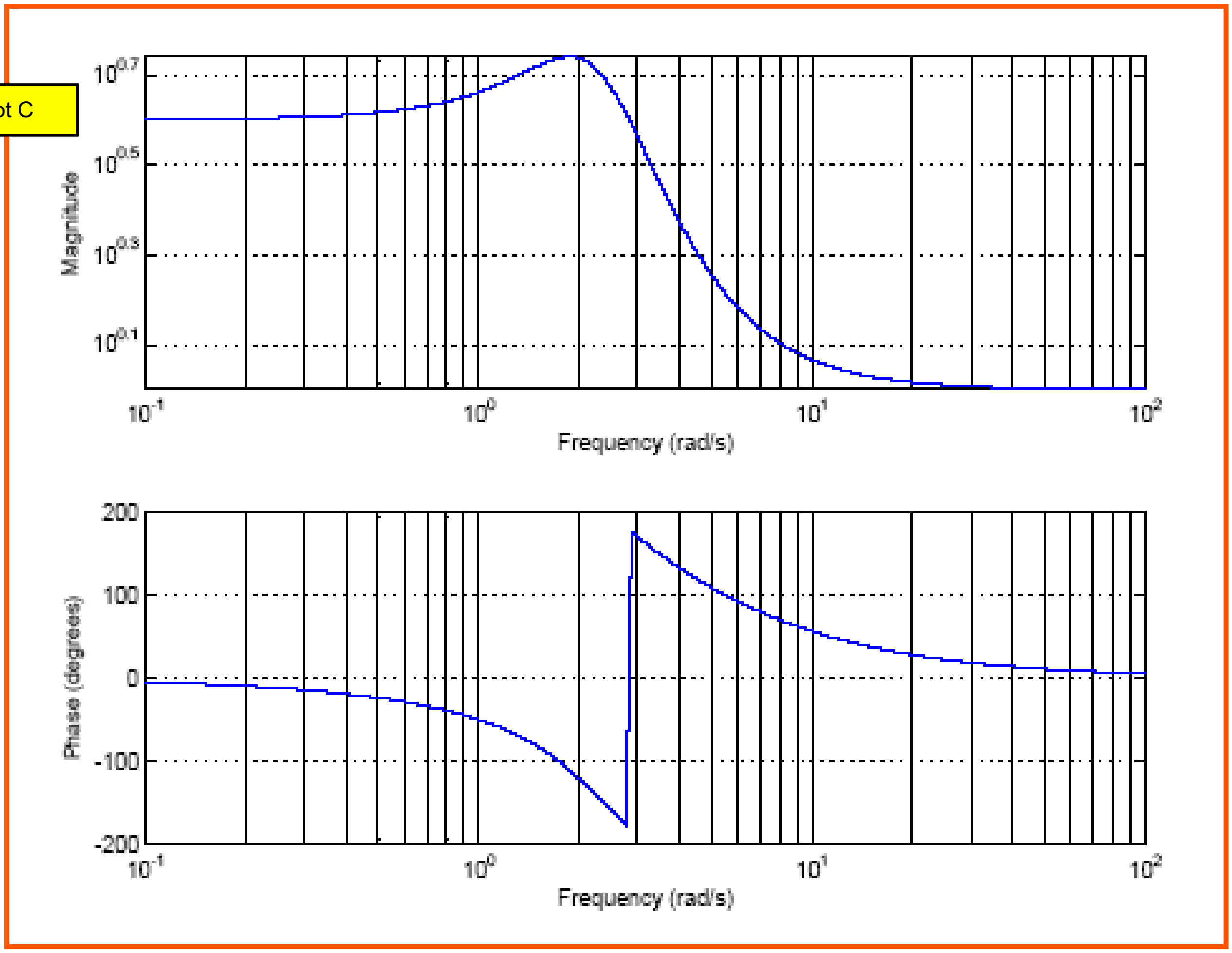
Magnitude
$10^{0.3}$
$10^{0.2}$
$10^{0.1}$
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)
Phase (degrees)
5
0
-5
-10
-15
-20
-25
-30
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)

Plot B



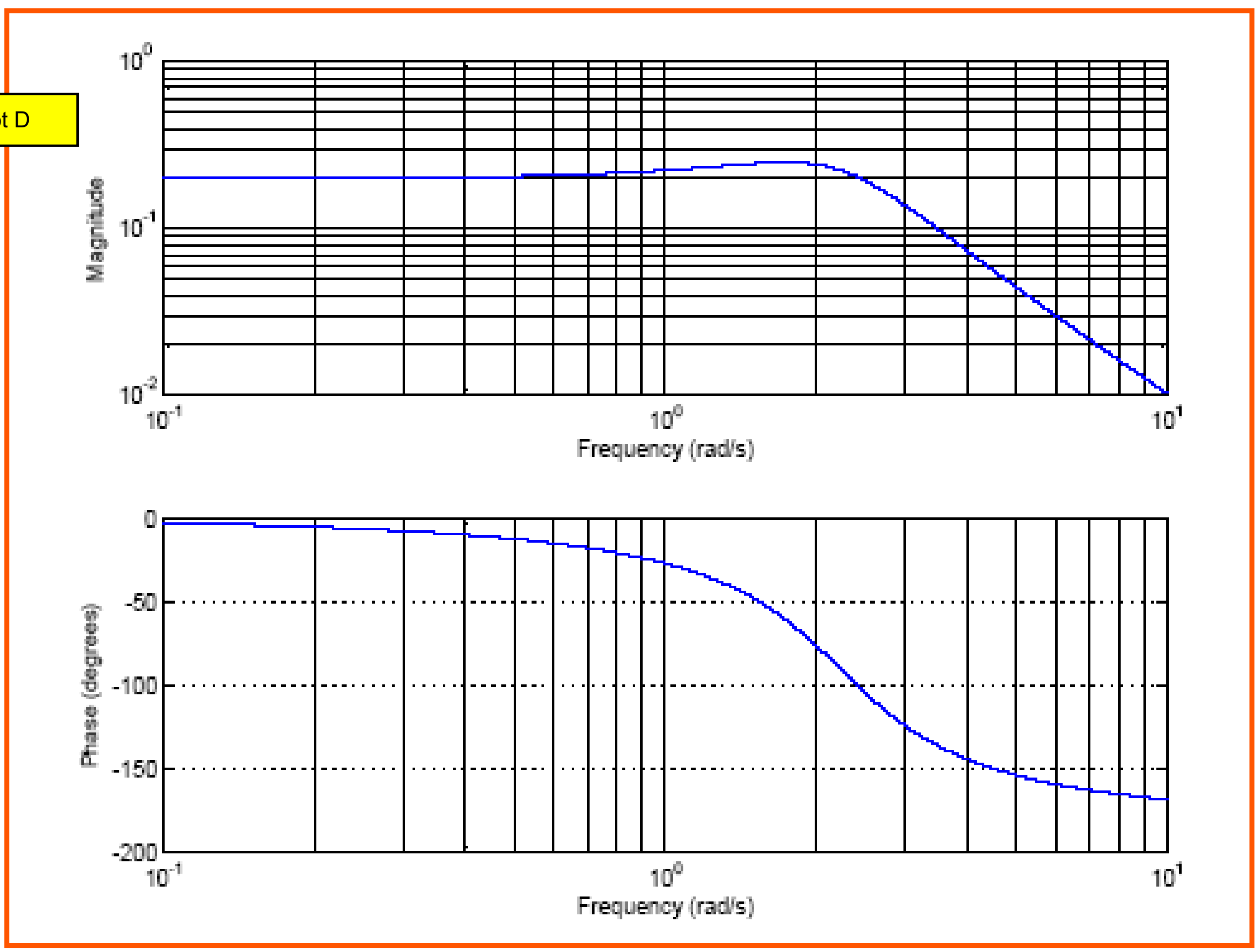
Magnitude
$10^{0.7}$
$10^{0.5}$
$10^{0.3}$
$10^{0.1}$
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)
Phase (degrees)
0
-10
-20
-30
-40
-50
-60
-70
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Plot C



Magnitude
$10^{0.7}$
$10^{0.5}$
$10^{0.3}$
$10^{0.1}$
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)
Phase (degrees)
200
100
0
-100
-200
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Plot D



Magnitude
10^{0}
10^{-1}
10^{-2}
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)
Phase (degrees)
0
-50
-100
-150
-200
10^{-1}
10^{0}
10^{1}
Frequency (rad/s)